Evolving Smooth Manifolds of Arbitrary Codimension in $\mathbf{R}^n$

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The Problem: The goal of this research is to develop a theoretical and numerical framework to represent and evolve smooth manifolds of arbitrary codimension. By extending level set methods, our approach aims to address efficiently certain geometric variational problems arising in Computer Vision. Original ideas on the level set formulation of active contours need to be adapted to represent patterns that lie in arbitrary codimension manifolds of high dimensional feature spaces. Handling manifold boundaries as well as topological and dimensional changes are some of the issues considered in this work.

Motivation: Methods based on the evolution of curves and surfaces are gaining increasing attention in Computer Vision. In a classical image analysis setting, a curve is considered as embedded in an image and submitted to a force field created by elements of this image, e.g. edges and corners. Successful applications to image segmentation and contour tracking have raised expectations for more advanced tools, allowing open curves (or surfaces with boundaries) and changes of dimension. Moreover, beyond the image analysis issue, one would like to apply these ideas to the general data representation problem in feature spaces of dimension $n > 3$.

Previous Work: Using implicit representations via level set models [3], practical and efficient numerical methods have been developped for evolving planar curves and surfaces. At the same time, new theoretical developments [4] have provided some building blocks to extend these techniques to higher codimension problems. Some successful experiments have been conducted in codimension 2 for medical image segmentation [1], but no generic theoretical and numerical framework is available. Addressing this problem is the key idea of our work.

Approach: Formally, we are looking for a smooth manifold $\mathcal{M}$ of codimension k in $\mathbf{R}^n$ which minimizes a given cost functional. We assume $\mathcal{M}$ is the image of a single parametric patch $\mathcal{M} : \mathbf{R}^{n-k} \times [0, T) \longmapsto \mathbf{R}^n$. By computing the first variation of the functional, one can look for a suboptimal solution by building a gradient flow from an inital guess $\mathcal{M}_0$. In several cases, functionals can be simplified and the minimization process is reformulated as finding a minimal map in a suitable Riemannian space. For functionals related to image segmentation problems, the Riemannian metric is induced by image contours. This framework leads to the following type of initial value problems :

$$(1) \quad \begin{cases} \mathcal{M}_t(\mathbf{p}, t) & = & \mathbf{v} \\ \mathcal{M}(\cdot, 0) & = & \mathcal{M}_0 \end{cases}$$

where the velocity field $\mathbf{v}$ is piecewise smooth. Instead of dealing with this parametric representation, we propose an implicit description for the family $\mathcal{M}(\cdot, t)$. It relies on an auxiliary mapping $\mathbf{u} : \mathbf{R}^n \times [0, T) \longmapsto \mathbf{R}^n$ such that $\mathbf{u}(\mathcal{M}(\mathbf{p}, t), t) = \mathbf{0}$. This mapping, the so-called *Vector Distance Function (VDF)* [2], is defined as the gradient of the squared distance function to a given manifold. One can show that the family of solutions of (1) can be obtained implicitly by considering the following system of partial differential equations :

$$(2) \quad \begin{cases} \mathbf{u}_t(\mathbf{x}, t) & = & -\mathbf{v}_{\text{ext}}(\mathbf{x}) \\ \mathbf{u}(\cdot, 0) & = & \mathbf{u}_0 \end{cases}$$

At any time $t \in [0, T)$, $\mathbf{v}_{\text{ext}}$ satisfies a quasilinear boundary value PDE where $\mathbf{v}_{\text{ext}}$ is given by the projection of $\mathbf{v}$ on the normal space $\mathcal{N}_{\mathcal{M}(\mathbf{p})}$ for all points on $\mathcal{M}$. The evolution is designed such that $\mathbf{u}$ remains the gradient of the squared distance function to $\mathcal{M}(\cdot, t)$.

Difficulty: Preliminary numerical experiments (see Fig. 1 - 2) provide promising results on simple examples. Nevertheless, the design of stable and generic numerical schemes requires additional developments. The main issue is to cope efficiently with the nonlinearity of the underlying problem. Moreover, computational costs of high dimensional problems have to be carefully taken into account. We believe that significant improvements can be achieved by using multiresolution methods with coarse to fine dicretizations of the ambient space.

Future Work: Next steps include the validation of this approach with complex velocity fields borrowed from image segmentation models. Our final goal is the development of data representation methods for feature spaces of large dimensions. Applications to clustering and classification problems are considered.

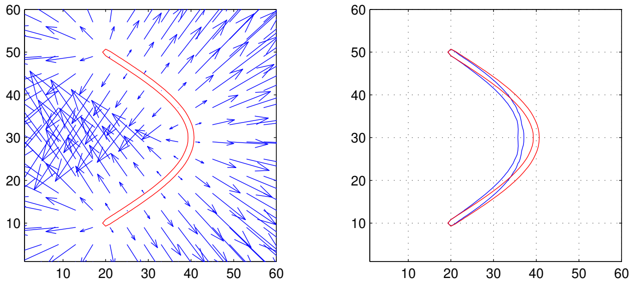

Figure 1: Left : A curve with boundary and its Vector Distance Function. Right : Resulting curve after a few iterations of the PDE on $\mathbf{u}$ with a discontinuous velocity field (mean curvature flow for $\mathcal{M} \setminus \partial\mathcal{M}$ and zero speed on $\partial\mathcal{M}$)

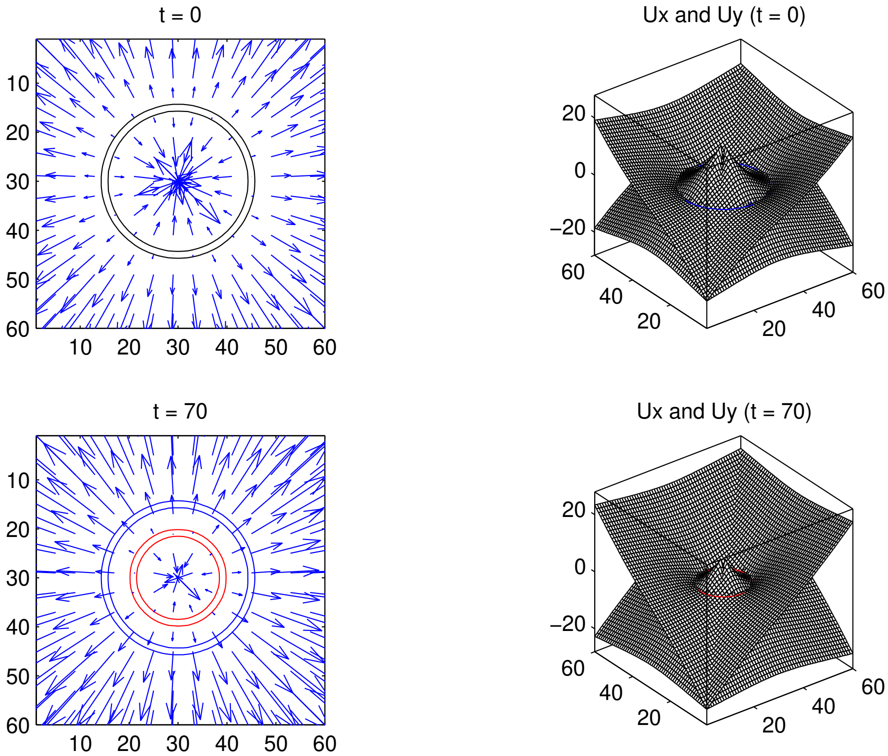

Figure 2: Circle in $\mathbf{R}^2$ evolving under mean curvature flow. Left : The evolving circle and its Vector Distance Function (presented as a vector field). Right : The two components of the Vector Distance Function.

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