6.801/866

Projective Structure from Motion

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Administrivia

Pset 4 delayed one day; on web tomorrow.

Last lecture readings: 22

Today's F&P readings: 13.0, 13.1, 13.4, 13.5

Today:

- Projective spaces
- Cross ratio
- Geometric reconstruction
- Factorization algorithm
- Euclidean upgrade

Projective SFM approach

Ignoring at first the Euclidean constraints associated with calibrated cameras will linearize the recovery of scene structure and camera motion from point correspondences

Decompose motion analysis into two stages

1. recovery of the projective shape of the scene and the estimation of the corresponding projection matrices.
2. exploit the geometric constraints associated with (partially or fully) calibrated perspective cameras to upgrade the projective reconstruction to a Euclidean one.

Review: Perspective Projection

$$\boldsymbol{p} = \frac{1}{z}\mathcal{M}\boldsymbol{P}, \quad \text{where} \quad \mathcal{M} = \mathcal{K}\begin{pmatrix}\mathcal{R} & \boldsymbol{t}\end{pmatrix}$$

or

$$u = \frac{\boldsymbol{m}_1 \cdot \boldsymbol{P}}{\boldsymbol{m}_3 \cdot \boldsymbol{P}}$$
$$v = \frac{\boldsymbol{m}_2 \cdot \boldsymbol{P}}{\boldsymbol{m}_3 \cdot \boldsymbol{P}}$$

where $\boldsymbol{m}_{i1}^T$, $\boldsymbol{m}_{i2}^T$ and $\boldsymbol{m}_{i3}^T$ denote the rows of the 3 × 4 projection matrix M

Projective SFM

Goal: Estimate M and P from (u_{ij}, v_{ij})…

$$u_{ij} = \frac{\boldsymbol{m}_{i1} \cdot \boldsymbol{P}_j}{\boldsymbol{m}_{i3} \cdot \boldsymbol{P}_j} \qquad \text{for} \quad i = 1, \ldots, m \quad \text{and} \quad j = 1, \ldots, n,$$
$$v_{ij} = \frac{\boldsymbol{m}_{i2} \cdot \boldsymbol{P}_j}{\boldsymbol{m}_{i3} \cdot \boldsymbol{P}_j}$$

Projective Ambiguity

if $\boldsymbol{P}_j$ and $\mathcal{M}_i$ are solutions to the SFM equations, then so are

$$\mathcal{M}'_i = \mathcal{M}_i Q$$

$$\boldsymbol{P}'_j = Q^{-1}\boldsymbol{P}_j$$

where Q is a projective transformation matrix (arbitrary nonsingular 4x4 matrix, defined up to scale)

Projective Geometry

The means of measurement available in projective geometry are even more primitive than those available in affine geometry

- no notions of lengths, areas and angles (Euclidean)
- no notions of ratios of lengths along parallel lines (Affine)
- no notion of parallelism (Affine)

The concepts of points, lines and planes remain.

And a new, weaker scalar measure of the arrangement of collinear points, the cross-ratio…

The Cross-ratio

The non-homogeneous projective coordinates of a point can be defined geometrically in terms of cross-ratios.

Given four collinear points A, B, C, D such that A, B and C are distinct, we define the cross-ratio of these points as:

$$\{A, B; C, D\} \stackrel{\text{def}}{=} \frac{\overline{CA}}{\overline{CB}} \times \frac{\overline{DB}}{\overline{DA}}$$

$$\{A, B; C, D\} \stackrel{\text{def}}{=} \frac{\overline{CA}}{\overline{CB}} \times \frac{\overline{DB}}{\overline{DA}}$$

The value of this cross ratio is independent of the intersecting line or plane:

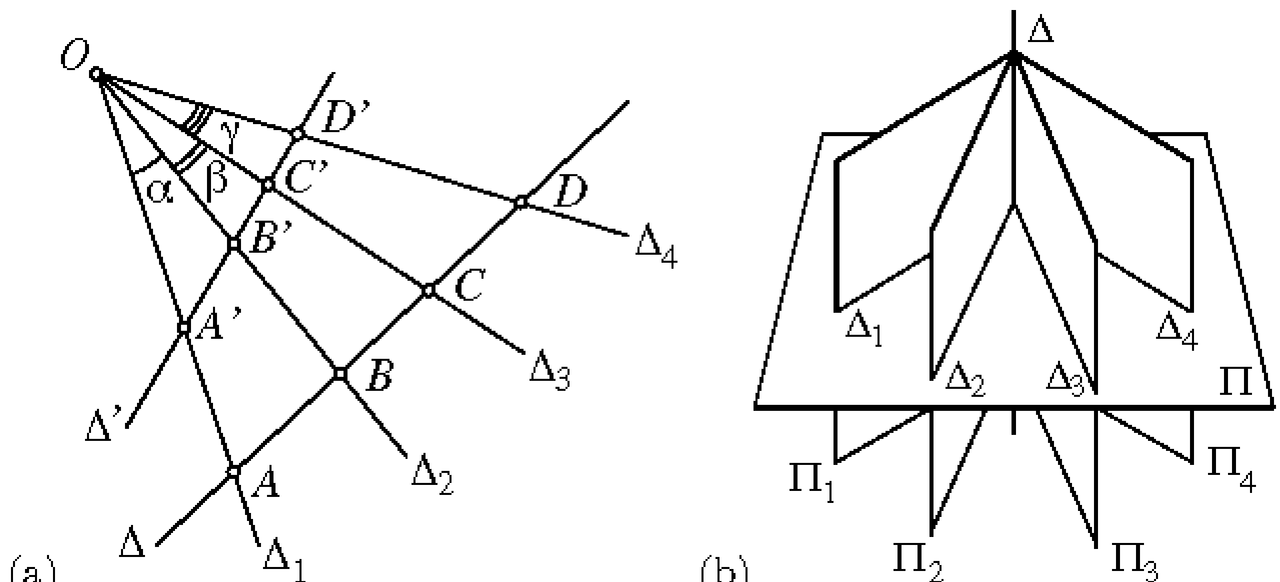
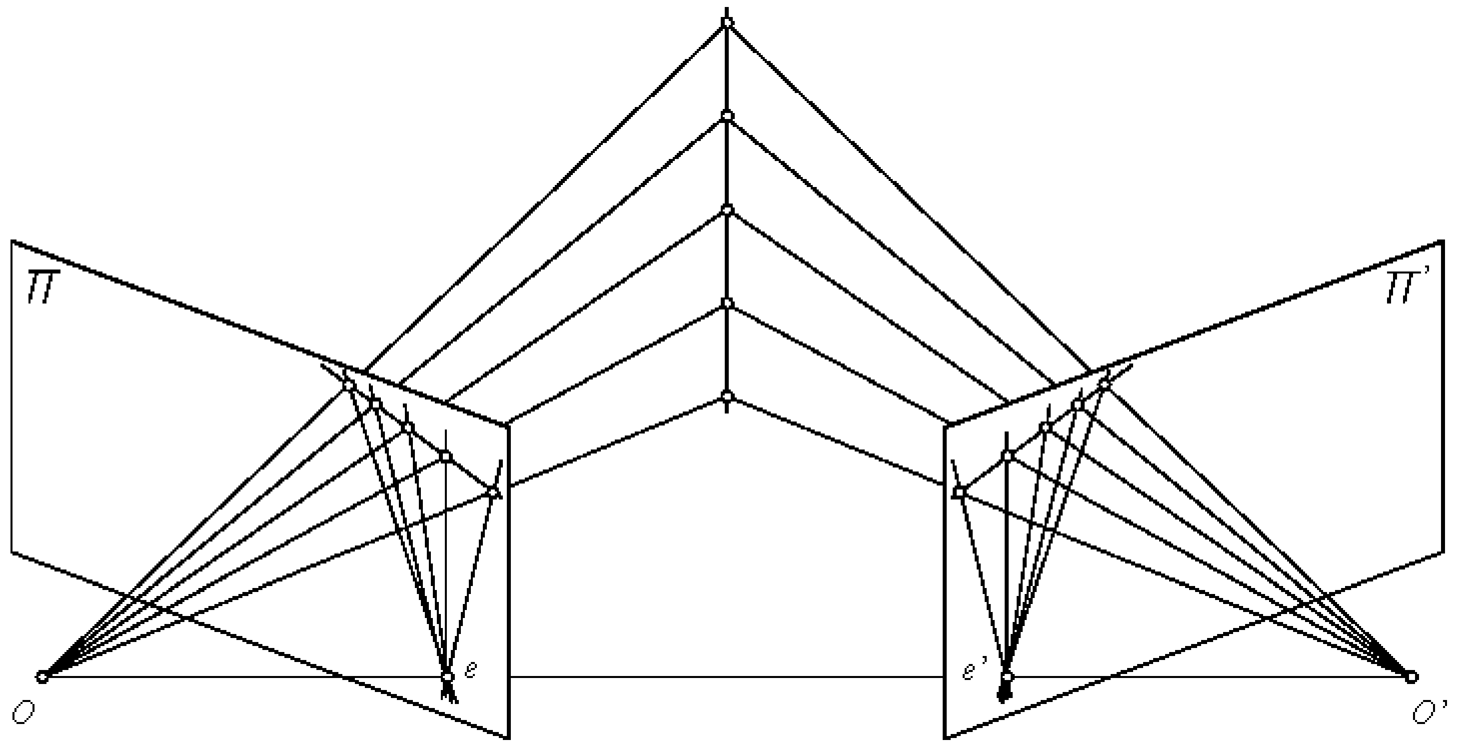


FIGURE 14.2: Definition of the cross-ratio of: (a) four lines and (b) four planes. As shown in the exercises, the cross-ratio $\{A, B; C, D\}$ depends only on the three angles α, β and γ. In particular, we have $\{A, B; C, D\} = \{A', B'; C', D'\}$.

Epipolar Transformation is Projective

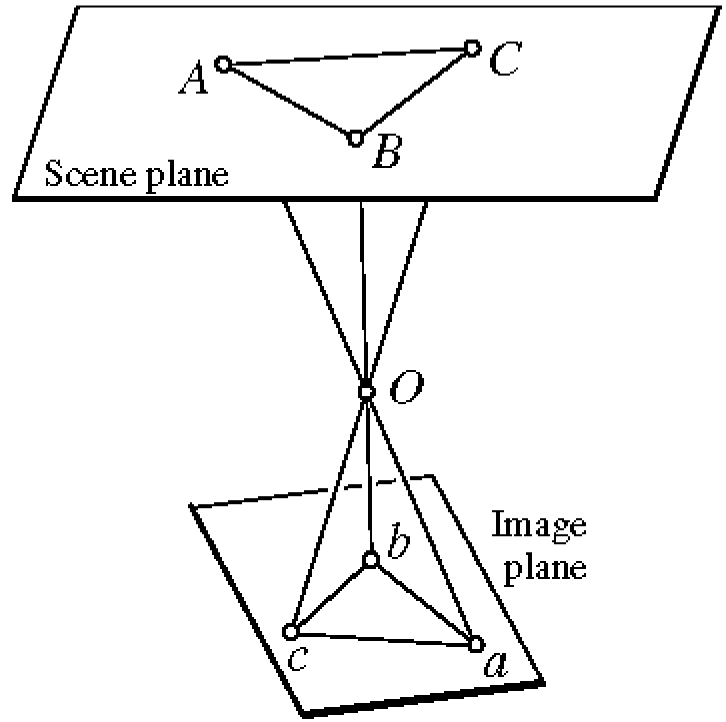


Cross-ratio of any quadruple of epipolar lines is invariant.

Thus epipolar transform is projective and is a Homography.

Homography

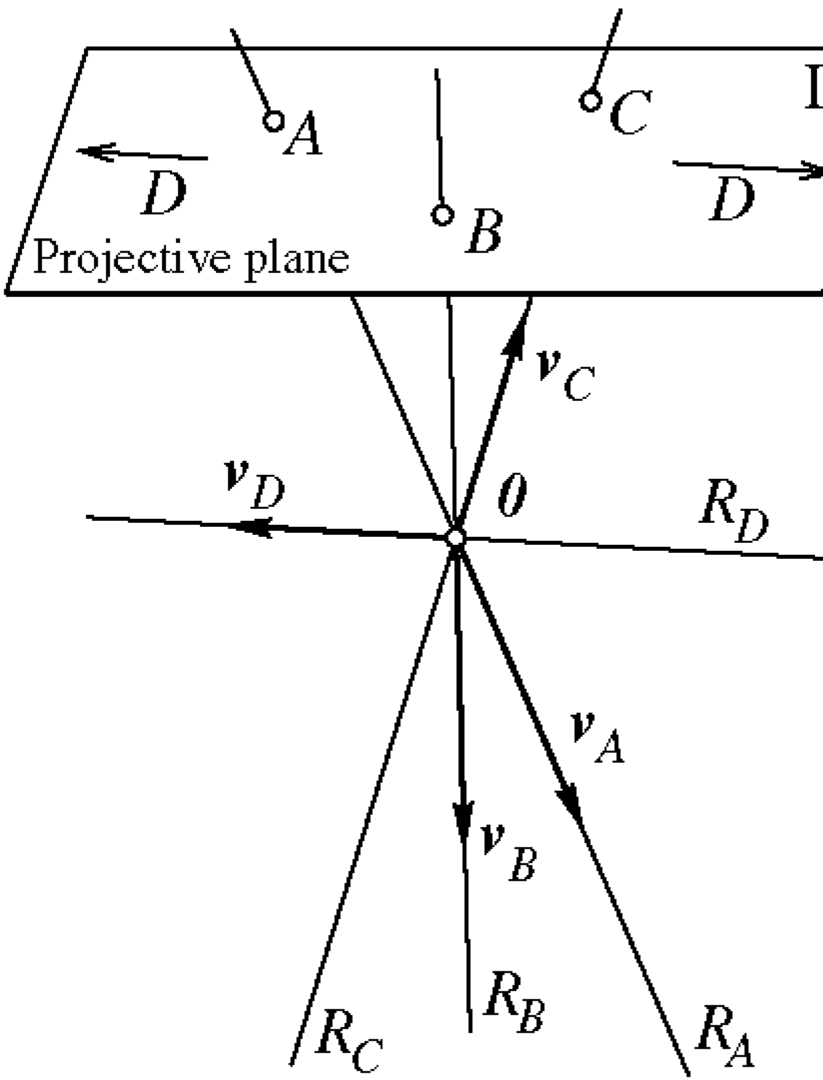
Projective transform:
bijective linear map
a.k.a. *Homography*



Consider two planes and a point O lying outside these planes in $\mathcal{E}^3$.

The perspective projection mapping any point A in the (projective closure of the) first plane onto the intersection of the line AO with the (projective closure of the) second plane is a projective transformation.

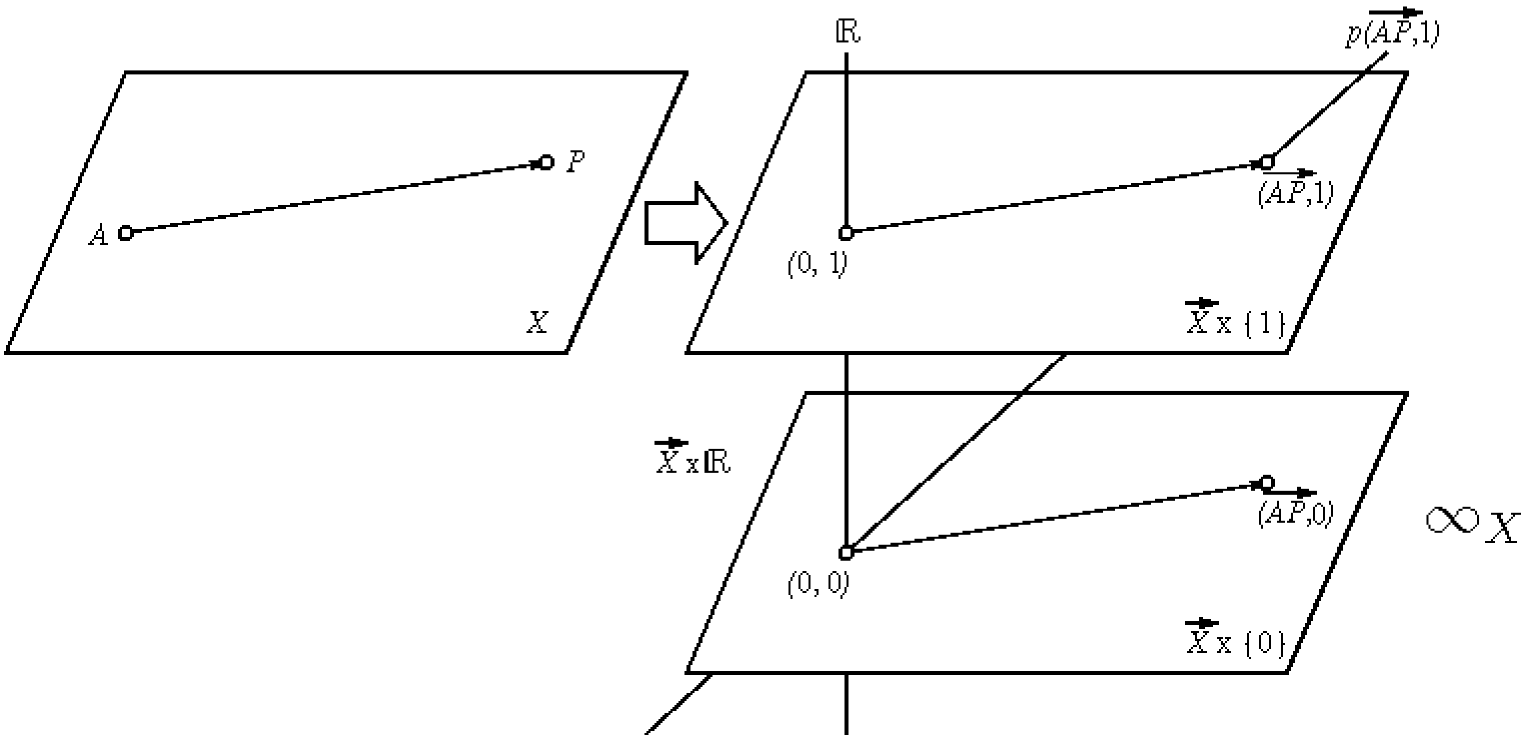
Projective Geometry



an affine plane Π of $\mathbb{R}^3$

- Rays R_A, R_B and R_C associated with the vectors v_A, v_B and v_C below can be mapped onto the points A,B,C
- The vectors v_A, v_B and v_C are linearly independent, and thus so are the points A,B,C
- As a ray becomes close to parallel to Π the point where it intersects Π moves to infinity
- Projective plane can be modeled by adding set of *points at infinity* to 2-D Π

Affine plane embedded in projective space



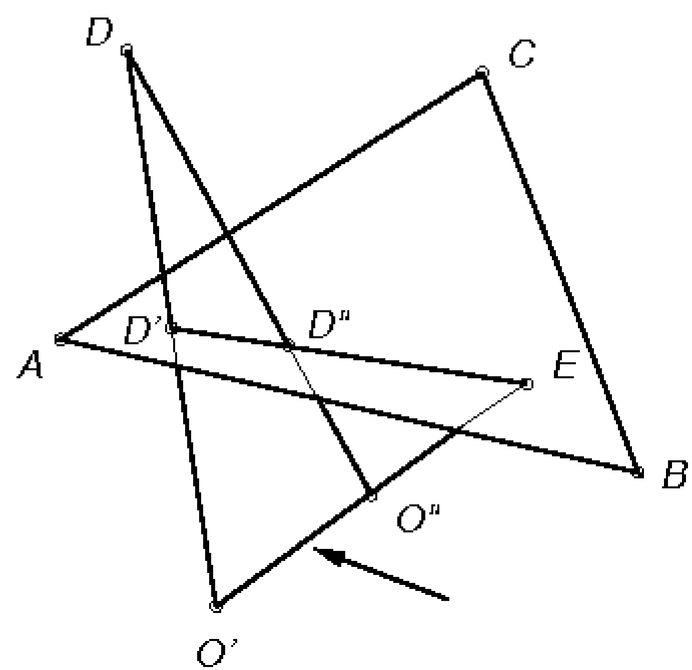
Add (0,1) to affine plane $\tilde{X} \stackrel{\text{def}}{=} P(\vec{X} \times \mathbb{R})$

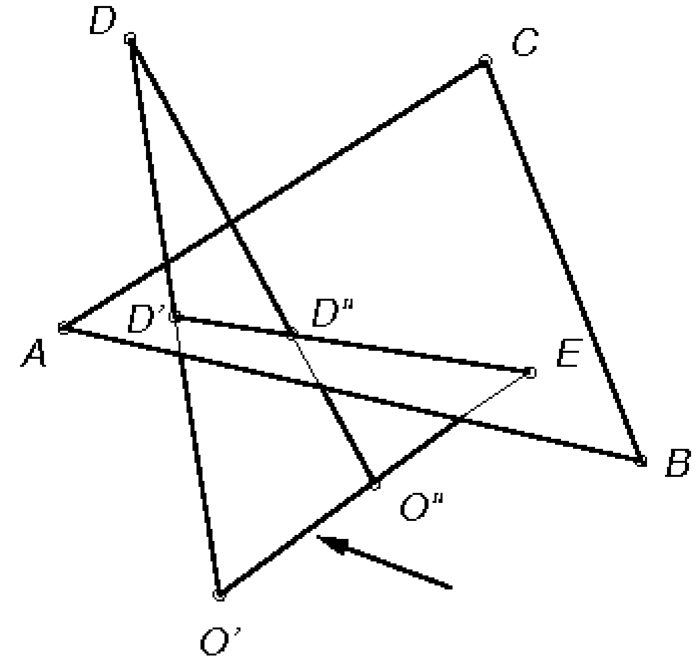
to represent the *hyperplane at infinity* ∞X

This projective completion justifies the notion of homogeneous coordinates.

Geometric reconstruction

Geometric construction of the projective coordinates of the point D in the basis formed by the five points A, B, C, O and O .

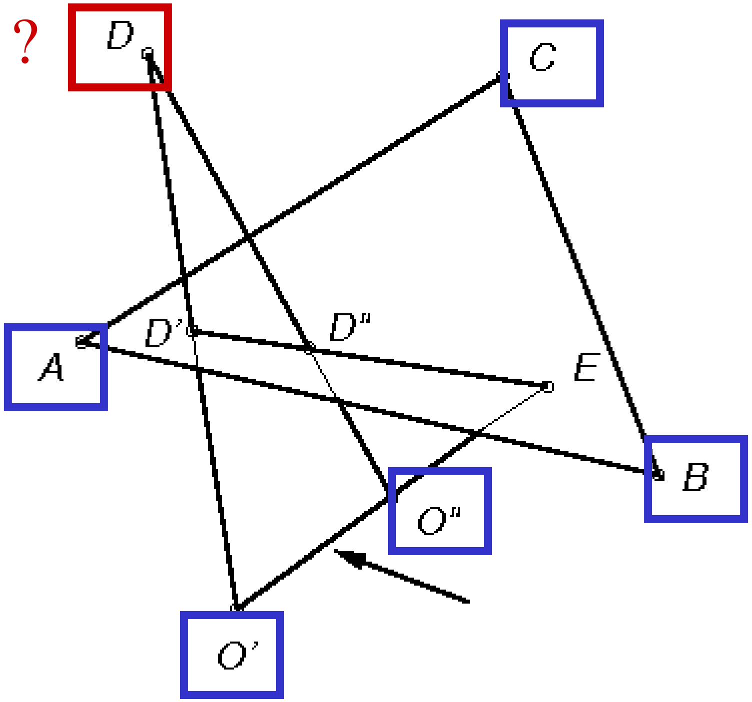




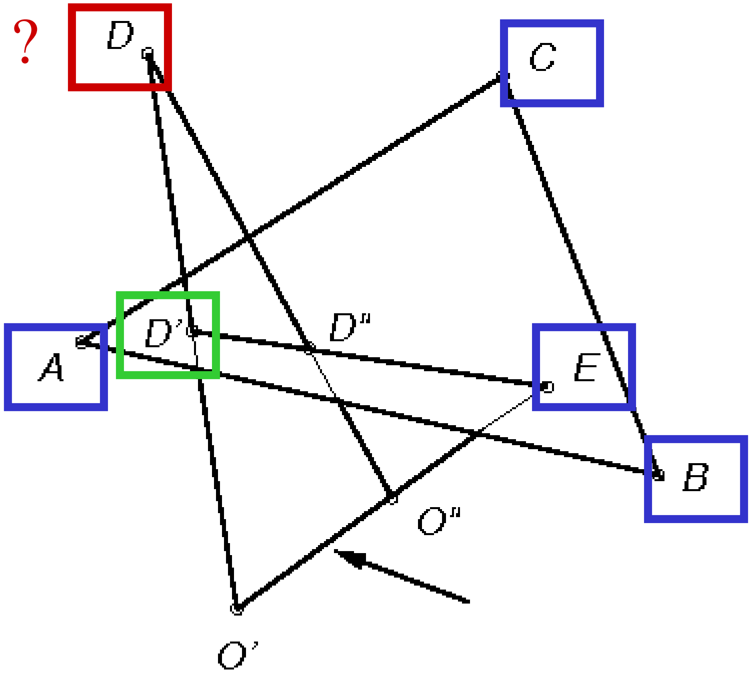
Observe four non-coplanar points A, B, C, D with a weakly-calibrated stereo rig. Let O' / O'' denote the position of the optical center of the first / second camera. Let:

- P' be the intersection of the ray $O'P$ with the plane ABC
- P'' be the intersection of the ray $O''P$ with the plane ABC
- p' be projection of P into the first image
- p'' be projection of P into the second image

The epipoles are e' and e'' and the baseline intersects the plane ABC in E. (Clearly, in projective coordinates $E' = E'' = E$, $A' = A'' = A$, etc.)

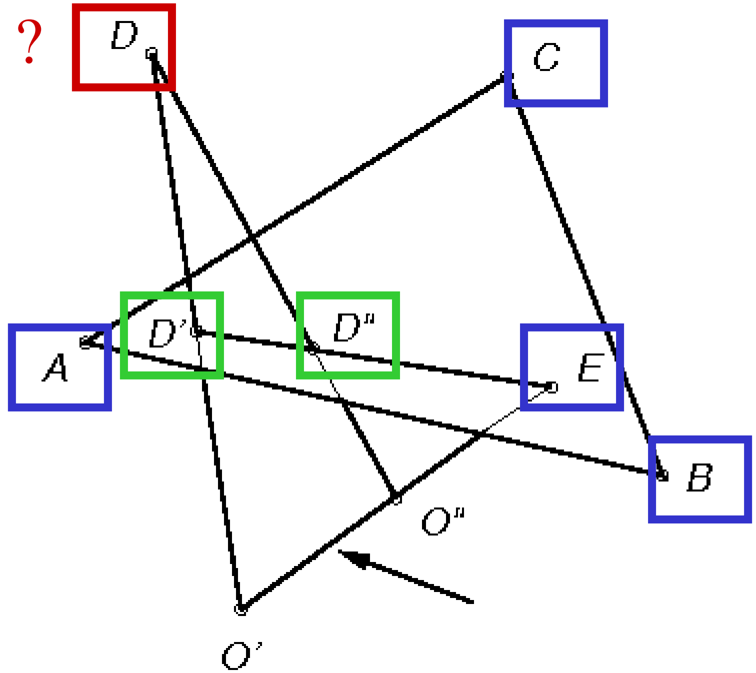


We choose *A*‹*B*‹*C*‹*O*’‹*O*” as a basis for projective three-space, and our goal is to reconstruct the position of *D*.



We choose A‹B‹C‹O'‹O'' as a basis for projective three-space, and our goal is to reconstruct the position of D.

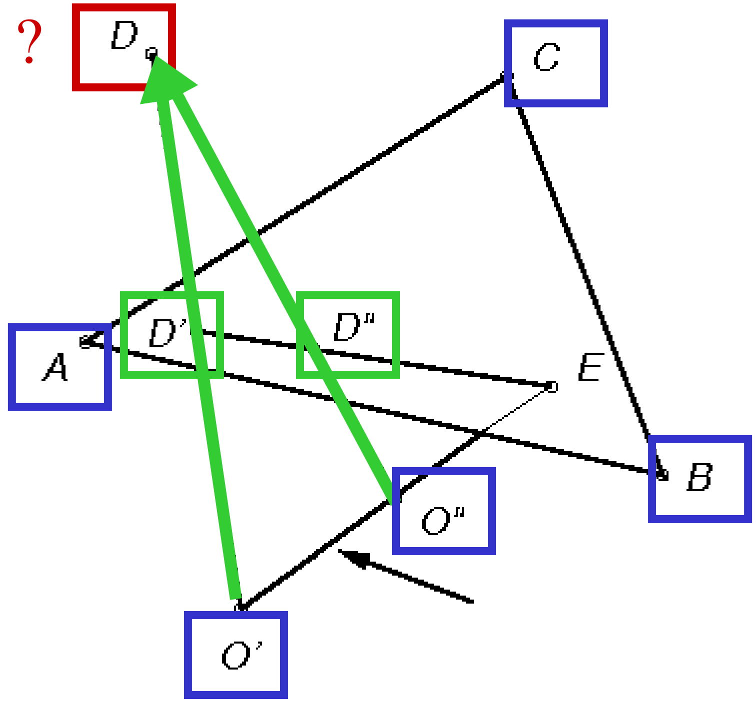
Choosing a'‹b'‹c'‹e' as a basis for the first image plane, we can measure the coordinates of d' in this basis and reconstruct the point D' in the basis A‹B‹C‹$E$of the plane ABC.



We choose *A*‹*B*‹*C*‹*O*'‹*O*" as a basis for projective three-space, and our goal is to reconstruct the position of *D*.

Choosing *a*'‹*b*'‹*c*'‹*e*' as a basis for the first image plane, we can measure the coordinates of *d*' in this basis and reconstruct the point *D*' in the basis *A*‹*B*‹*C*‹*E*of the plane *ABC*.

Similarly, we can reconstruct the point *D*" from the projective coordinates *d*' in the basis *a*"‹*b*"‹*c*"‹*e*" of the second image plane.

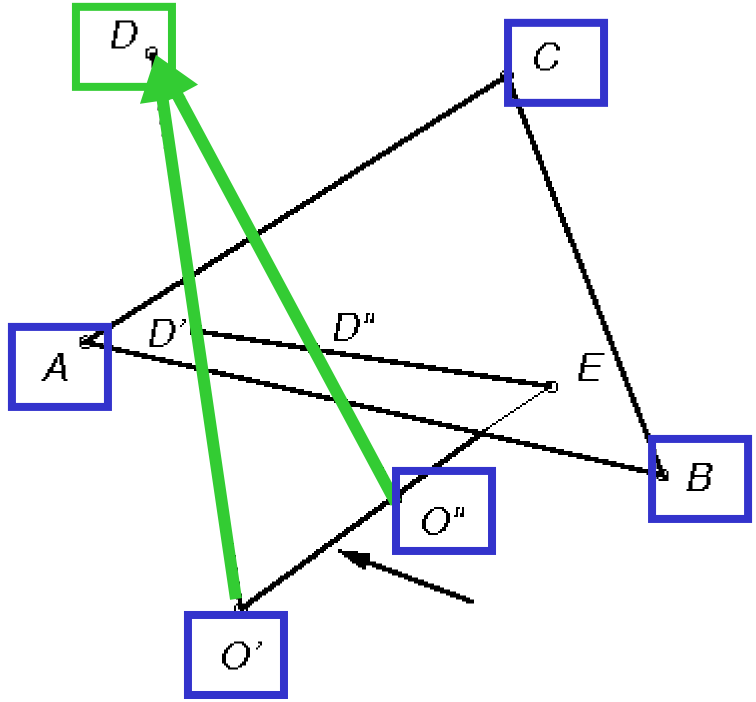


We choose A, B, C, O', O'' as a basis for projective three-space, and our goal is to reconstruct the position of D.

Choosing a', b', c', e' as a basis for the first image plane, we can measure the coordinates of d' in this basis and reconstruct the point D' in the basis A, B, C, $E$of the plane ABC.

Similarly, we can reconstruct the point D'' from the projective coordinates d' in the basis a'', b'', c'', e'' of the second image plane.

The point $D$is finally reconstructed as the intersection of the two lines OD and OD.



We choose A‹B‹C‹O'‹O'' as a basis for projective three-space, and our goal is to reconstruct the position of D.

Choosing a'‹b'‹c'‹e' as a basis for the first image plane, we can measure the coordinates of d' in this basis and reconstruct the point D' in the basis A‹B‹C‹$E$of the plane ABC.

Similarly, we can reconstruct the point D'' from the projective coordinates d' in the basis a''‹b''‹c''‹e'' of the second image plane.

The point $D$is finally reconstructed as the intersection of the two lines OD and OD.

We can now express this geometric construction in algebraic terms. It turns out to be simpler to reorder the points of our projective frame and to calculate the non-homogeneous projective coordinates of *D*in the basis formed by the tetrahedron *A*‹*O*‹*O*‹*B*and the unit point *C*. These coordinates are defined by the following three cross-ratios:

$$
\begin{aligned}
k_0 &= \{O''O'A,\ O''O'B;\ O''O'C,\ O''O'D\},\\
k_1 &= \{O'AO'',\ O'AB;\ O'AC,\ O'AD\},\\
k_2 &= \{AO''O',\ AO''B;\ AO''C,\ AO''D\}.
\end{aligned}
$$

By intersecting the corresponding pencils of planes with the two image planes we immediately obtain the values of *k*0‹*k*1‹*k*2 as cross-ratios directly measurable in the two images:

$$
\begin{aligned}
k_0 &= \{e'a',\ e'b';\ e'c',\ e'd'\} = \{e''a'',\ e''b'';\ e''c'',\ e''d''\},\\
k_1 &= \{a'e',\ a'b';\ a'c',\ a'd'\},\\
k_2 &= \{a''e'',\ a''b'';\ a''c'',\ a''d''\}.
\end{aligned}
$$

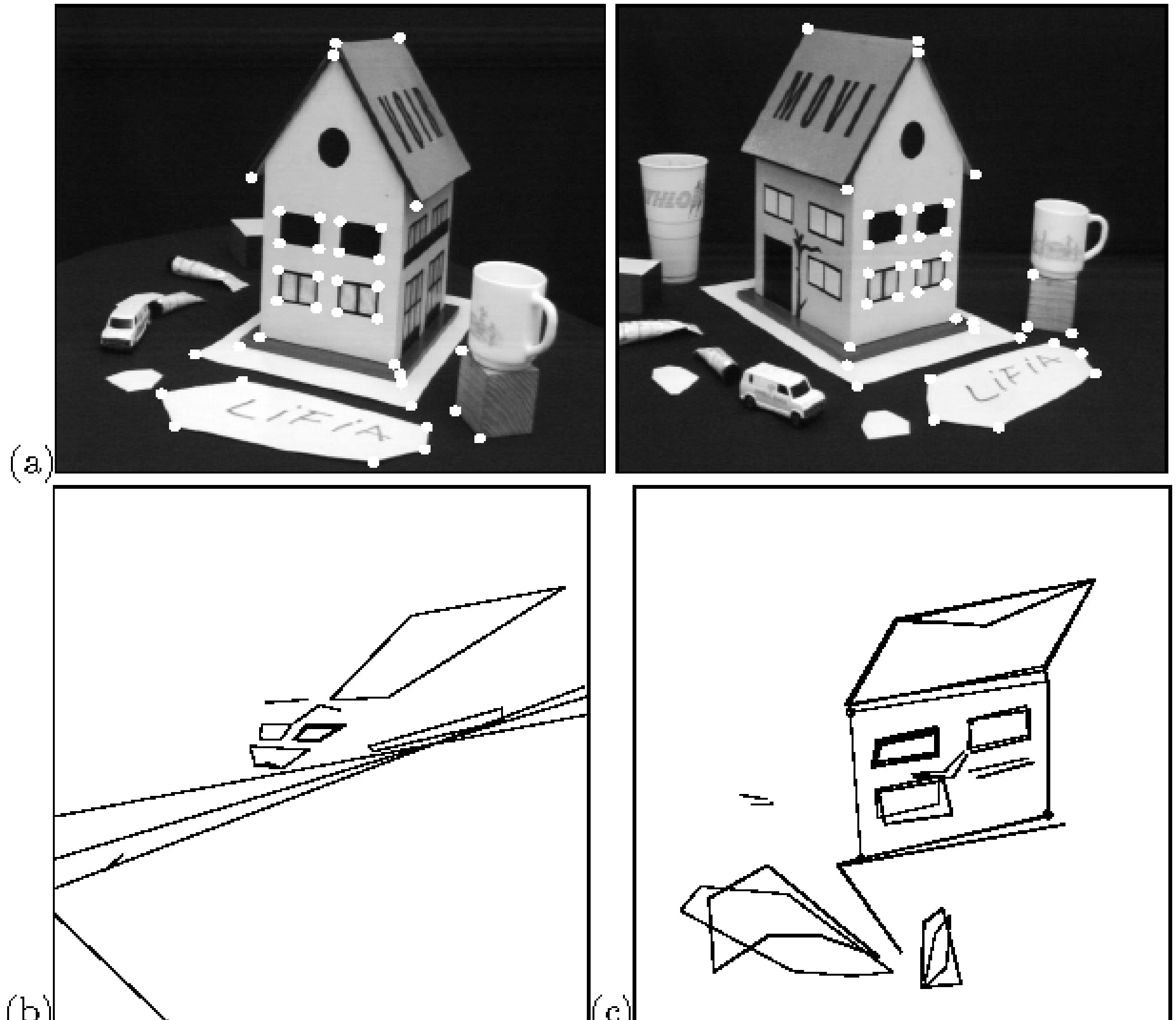


FIGURE 14.4: Geometric point reconstruction: (a) input data; (b) raw projective coordinates; (c) corrected projective coordinates. Reprinted from [Ponce *et al.*, 1993], Figures 1 and 9.

Factorization approach to Projective SFM

Use multiple frame sequence….

Generalize Tomasi-Kanade to the projective case…

Given m images of n points, we can express the data matrix as

$$\mathcal{D} = \mathcal{M}\mathcal{P},$$

with

$$\mathcal{D} \stackrel{\text{def}}{=} \begin{pmatrix} z_{11}\boldsymbol{p}_{11} & z_{12}\boldsymbol{p}_{12} & \dots & z_{1n}\boldsymbol{p}_{1n} \\ z_{21}\boldsymbol{p}_{21} & z_{22}\boldsymbol{p}_{22} & \dots & z_{2n}\boldsymbol{p}_{2n} \\ \dots & \dots & \dots & \dots \\ z_{m1}\boldsymbol{p}_{m1} & z_{m2}\boldsymbol{p}_{m2} & \dots & z_{mn}\boldsymbol{p}_{mn} \end{pmatrix}, \ \mathcal{M} \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \dots \\ \mathcal{M}_m \end{pmatrix} \text{ and } \mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2 & \dots & \boldsymbol{P}_n \end{pmatrix}.$$

Factorization approach to Projective SFM

$$\mathcal{D} = \mathcal{M}\mathcal{P}, \qquad \mathcal{D} \stackrel{\text{def}}{=} \begin{pmatrix} z_{11}\boldsymbol{p}_{11} & z_{12}\boldsymbol{p}_{12} & \dots & z_{1n}\boldsymbol{p}_{1n} \\ z_{21}\boldsymbol{p}_{21} & z_{22}\boldsymbol{p}_{22} & \dots & z_{2n}\boldsymbol{p}_{2n} \\ \dots & \dots & \dots & \dots \\ z_{m1}\boldsymbol{p}_{m1} & z_{m2}\boldsymbol{p}_{m2} & \dots & z_{mn}\boldsymbol{p}_{mn} \end{pmatrix}, \ \mathcal{M} \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \dots \\ \mathcal{M}_m \end{pmatrix} \text{ and } \mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2 & \dots & \boldsymbol{P}_n \end{pmatrix}.$$

As the product of $3m \times 4$ and $4 \times n$ matrices, the $3m \times n$ matrix $\mathcal{D}$ has (at most) rank 4

If the projective depths z_{ij} were known, we could compute $\mathcal{M}$ and $\mathcal{P}$, by using singular value decomposition to factor $\mathcal{D}$.

If $\mathcal{M}$ and $\mathcal{P}$ were known, we can solve directly for the values of the projective depths.

Iterative approach

$$\mathcal{D} = \mathcal{M}\mathcal{P}, \qquad \mathcal{D} \stackrel{\text{def}}{=} \begin{pmatrix} z_{11}\boldsymbol{p}_{11} & z_{12}\boldsymbol{p}_{12} & \dots & z_{1n}\boldsymbol{p}_{1n} \\ z_{21}\boldsymbol{p}_{21} & z_{22}\boldsymbol{p}_{22} & \dots & z_{2n}\boldsymbol{p}_{2n} \\ \dots & \dots & \dots & \dots \\ z_{m1}\boldsymbol{p}_{m1} & z_{m2}\boldsymbol{p}_{m2} & \dots & z_{mn}\boldsymbol{p}_{mn} \end{pmatrix}, \ \mathcal{M} \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \dots \\ \mathcal{M}_m \end{pmatrix} \text{ and } \mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2 & \dots & \boldsymbol{P}_n \end{pmatrix}.$$

This suggests an iterative scheme for estimating the unknowns z_{ij} , $\mathcal{M}$ and $\mathcal{P}$ by alternating steps where some of these unknowns are held constant while others are estimated.

- Assume projective depths z_{ij} are known, and compute $\mathcal{M}$ and $\mathcal{P}$, using singular value decomposition to factor $\mathcal{D}$.
- Assume $\mathcal{M}$ and $\mathcal{P}$ ⊣∇⌉ \≀⊒ ⌊≀∇∇⌉⌋⊔ ⊣\⌈ solve directly for the values of the projective depths.

Iterative approach

$$\mathcal{D} = \mathcal{M}\mathcal{P}, \qquad \mathcal{D} \stackrel{\text{def}}{=} \begin{pmatrix} z_{11}\boldsymbol{p}_{11} & z_{12}\boldsymbol{p}_{12} & \dots & z_{1n}\boldsymbol{p}_{1n} \\ z_{21}\boldsymbol{p}_{21} & z_{22}\boldsymbol{p}_{22} & \dots & z_{2n}\boldsymbol{p}_{2n} \\ \dots & \dots & \dots & \dots \\ z_{m1}\boldsymbol{p}_{m1} & z_{m2}\boldsymbol{p}_{m2} & \dots & z_{mn}\boldsymbol{p}_{mn} \end{pmatrix}, \ \mathcal{M} \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \dots \\ \mathcal{M}_m \end{pmatrix} \text{ and } \mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2 & \dots & \boldsymbol{P}_n \end{pmatrix}.$$

– Assume projective depths z_{ij} are known, and compute $\mathcal{M}$ and $\mathcal{P}$, using singular value decomposition to factor $\mathcal{D}$.

– Assume $\mathcal{M}$ and $\mathcal{P}$ ⊣∇⌉ \≀⊒ ⌋≀∇∇⌉⌋⊔ ⊣\⌈ solve directly for the values of the projective depths.

Iterative approach to Projective Factorization

$$\mathcal{D} = \mathcal{M}\mathcal{P}, \qquad \mathcal{D} \stackrel{\text{def}}{=} \begin{pmatrix} z_{11}\boldsymbol{p}_{11} & z_{12}\boldsymbol{p}_{12} & \dots & z_{1n}\boldsymbol{p}_{1n} \\ z_{21}\boldsymbol{p}_{21} & z_{22}\boldsymbol{p}_{22} & \dots & z_{2n}\boldsymbol{p}_{2n} \\ \dots & \dots & \dots & \dots \\ z_{m1}\boldsymbol{p}_{m1} & z_{m2}\boldsymbol{p}_{m2} & \dots & z_{mn}\boldsymbol{p}_{mn} \end{pmatrix}, \ \mathcal{M} \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \dots \\ \mathcal{M}_m \end{pmatrix} \text{ and } \mathcal{P} \stackrel{\text{def}}{=} (\boldsymbol{P}_1 \ \boldsymbol{P}_2 \ \dots \ \boldsymbol{P}_n).$$

– Assume projective depths z_{ij} are known, and compute $\mathcal{M}$ and $\mathcal{P}$, using singular value decomposition to factor $\mathcal{D}$.

– Assume $\mathcal{M}$ and $\mathcal{P}$ ⊣∇⌉ \≀⊒ ⌋≀∇∇⌉⌋⊔ ⊣\⌈ solve directly for the values of the projective depths.

…use Tomasi and Kanade SVD algorithm

Review: Affine case

With an appropriate choice of origin (e.g., first point, centriod),

$$\boldsymbol{p}_i = \mathcal{A}_i \boldsymbol{P} \qquad \boldsymbol{q} = \mathcal{A} \boldsymbol{P},$$

and the data matrix becomes:

$$\mathcal{D} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{q}_1 & \dots & \boldsymbol{q}_n \end{pmatrix} = \mathcal{A} \mathcal{P}$$

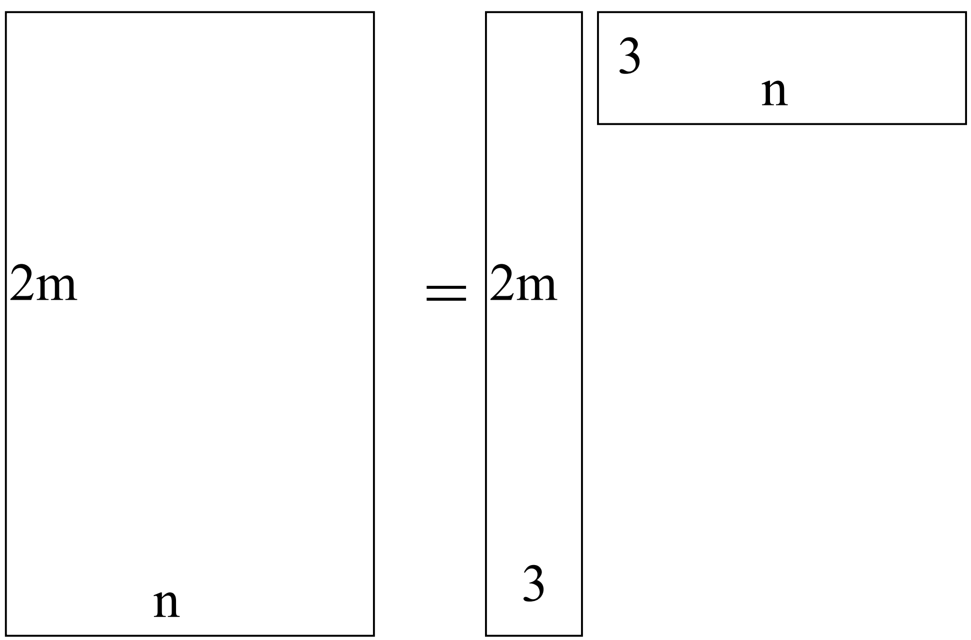
$$\mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{P}_1 & \dots & \boldsymbol{P}_n \end{pmatrix}.$$

Review: Affine case

D = A P

Data-Matrix = Affine-Motions x 3-d-Points

(2m x n) = (2m x 3) x (3 x n)



D is rank 3 in affine case

Review: Factorization algorithm

Given a data matrix,

find Motion (A) and Shape (P) matrices that generate that data…

Tomasi and Kanade Factorization algorithm (1992):

Use Singular Value Decomposition to factor D into appropriately sized A and P.

Review: SVD

Technique: Singular Value Decomposition Let $\mathcal{A}$ be an $m \times n$ matrix, with $m \geq n$, then $\mathcal{A}$ can always be written as

$$\mathcal{A} = \mathcal{U}\mathcal{W}\mathcal{V}^T,$$

where:

- $\mathcal{U}$ is an $m \times n$ column-orthogonal matrix, i.e., $\mathcal{U}^T\mathcal{U} = \mathrm{Id}_m$,
- $\mathcal{W}$ is a diagonal matrix whose diagonal entries w_i $(i = 1, \ldots, n)$ are the singular values of $\mathcal{A}$ with $w_1 \geq w_2 \geq \ldots \geq w_n \geq 0$,
- and $\mathcal{V}$ is an $n \times n$ orthogonal matrix, i.e., $\mathcal{V}^T\mathcal{V} = \mathcal{V}\mathcal{V}^T = \mathrm{Id}_n$.

The SVD of a matrix can also be used to characterize matrices that are rank-deficient: suppose that $\mathcal{A}$ has rank $p < n$, then the matrices $\mathcal{U}$, $\mathcal{W}$, and $\mathcal{V}$ can be written as

$$\mathcal{U} = \begin{array}{|c|c|}\hline \mathcal{U}_p & \mathcal{U}_{n-p} \\ \hline \end{array} \quad \mathcal{W} = \begin{array}{|c|c|}\hline \mathcal{W}_p & 0 \\ \hline 0 & 0 \\ \hline \end{array} \quad \text{and} \quad \mathcal{V}^T = \begin{array}{|c|}\hline \mathcal{V}_p^T \\ \hline \mathcal{V}_{n-p}^T \\ \hline \end{array},$$

Review: Affine Factorization algorithm

1. Compute the singular value decomposition $\mathcal{D} = \mathcal{U}\mathcal{W}\mathcal{V}^T$.
2. Construct the matrices $\mathcal{U}_3$, $\mathcal{V}_3$, and $\mathcal{W}_3$ formed by the three leftmost columns of the matrices $\mathcal{U}$ and $\mathcal{V}$, and the corresponding 3×3 sub-matrix of $\mathcal{W}$.
3. Define

$$\mathcal{A}_0 = \mathcal{U}_3 \quad \text{and} \quad \mathcal{P}_0 = \mathcal{W}_3\mathcal{V}_3^T;$$

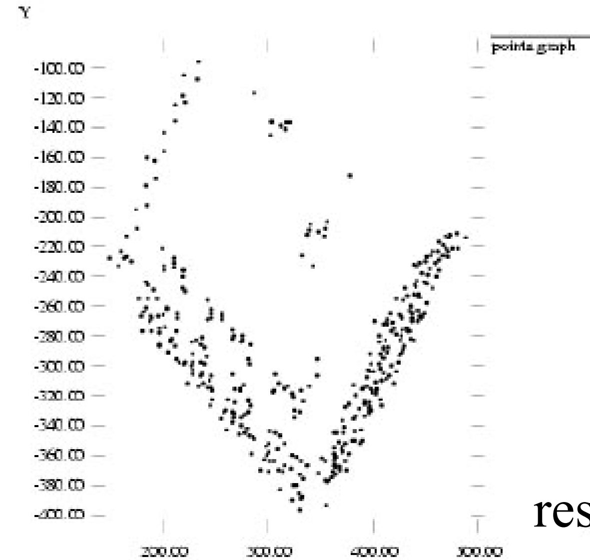
the $2m \times 3$ matrix $\mathcal{A}_0$ is an estimate of the camera motion, and the $3 \times n$ matrix $\mathcal{P}_0$ is an estimate of the scene structure.

Review: Affine Factorization algorithm

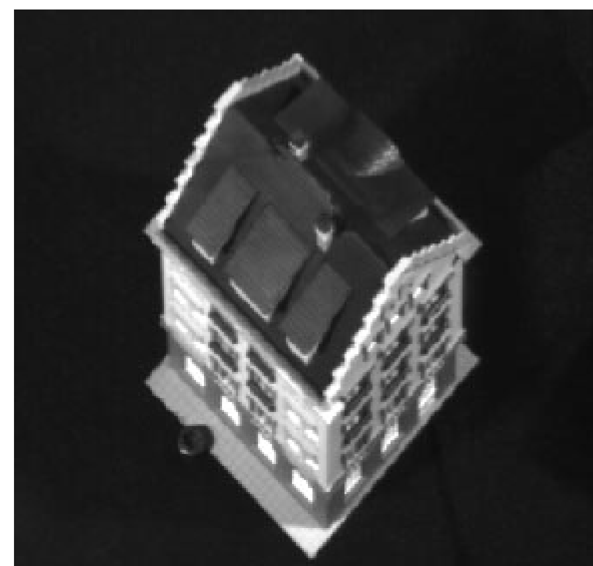
1

60

Input



result



comparision

Iterative approach to Projective Factorization

$$\mathcal{D} = \mathcal{M}\mathcal{P}, \qquad \mathcal{D} \stackrel{\text{def}}{=} \begin{pmatrix} z_{11}\boldsymbol{p}_{11} & z_{12}\boldsymbol{p}_{12} & \dots & z_{1n}\boldsymbol{p}_{1n} \\ z_{21}\boldsymbol{p}_{21} & z_{22}\boldsymbol{p}_{22} & \dots & z_{2n}\boldsymbol{p}_{2n} \\ \dots & \dots & \dots & \dots \\ z_{m1}\boldsymbol{p}_{m1} & z_{m2}\boldsymbol{p}_{m2} & \dots & z_{mn}\boldsymbol{p}_{mn} \end{pmatrix}, \; \mathcal{M} \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \dots \\ \mathcal{M}_m \end{pmatrix} \text{ and } \mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2 & \dots & \boldsymbol{P}_n \end{pmatrix}.$$

– Assume projective depths z_{ij} are known, and compute $\mathcal{M}$ and $\mathcal{P}$, using singular value decomposition to factor $\mathcal{D}$.

– Assume $\mathcal{M}$ and $\mathcal{P}$ ⊣∇⌉ \≀⊒ ⌋≀∇∇⌉⌋⊔ ⊣\⌈ solve directly for the values of the projective depths.

…use Tomasi and Kanade SVD algorithm

Iterative approach to Projective Factorization

$$\mathcal{D} = \mathcal{M}\mathcal{P}, \qquad \mathcal{D} \stackrel{\text{def}}{=} \begin{pmatrix} z_{11}\mathbf{p}_{11} & z_{12}\mathbf{p}_{12} & \dots & z_{1n}\mathbf{p}_{1n} \\ z_{21}\mathbf{p}_{21} & z_{22}\mathbf{p}_{22} & \dots & z_{2n}\mathbf{p}_{2n} \\ \dots & \dots & \dots & \dots \\ z_{m1}\mathbf{p}_{m1} & z_{m2}\mathbf{p}_{m2} & \dots & z_{mn}\mathbf{p}_{mn} \end{pmatrix}, \ \mathcal{M} \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \dots \\ \mathcal{M}_m \end{pmatrix} \text{ and } \mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2 & \dots & \boldsymbol{P}_n \end{pmatrix}.$$

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Find z_{ij} that best fit $\mathcal{P}$ and $\mathcal{M}$…

Iterative approach to Projective Factorization

$$\mathcal{D} = \mathcal{M}\mathcal{P}, \qquad \mathcal{D} \stackrel{\text{def}}{=} \begin{pmatrix} z_{11}\boldsymbol{p}_{11} & z_{12}\boldsymbol{p}_{12} & \dots & z_{1n}\boldsymbol{p}_{1n} \\ z_{21}\boldsymbol{p}_{21} & z_{22}\boldsymbol{p}_{22} & \dots & z_{2n}\boldsymbol{p}_{2n} \\ \dots & \dots & \dots & \dots \\ z_{m1}\boldsymbol{p}_{m1} & z_{m2}\boldsymbol{p}_{m2} & \dots & z_{mn}\boldsymbol{p}_{mn} \end{pmatrix}, \mathcal{M} \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \dots \\ \mathcal{M}_m \end{pmatrix} \text{ and } \mathcal{P} \stackrel{\text{def}}{=} (\boldsymbol{P}_1 \; \boldsymbol{P}_2 \; \dots \; \boldsymbol{P}_n).$$

– Assume projective depths z_{ij} are known, and compute $\mathcal{M}$ and $\mathcal{P}$, using singular value decomposition to factor $\mathcal{D}$.

– Assume $\mathcal{M}$ and $\mathcal{P}$ $\dashv\nabla\rceil$ $\setminus\wr\sqsupseteq$ $\lfloor\wr\nabla\nabla\rceil\rfloor\sqcup$ $\dashv\setminus\lceil$ solve directly for the values of the projective depths.

Find z_{ij} that best fit $\mathcal{P}$ and $\mathcal{M}$…

… express as solution to generalized eigenvalue problem

Find z_{ij} that best fit $\mathcal{P}$ and $\mathcal{M}$…

Goal:

$$E \stackrel{\text{def}}{=} |\mathcal{D} - \mathcal{M}\mathcal{P}|^2 = \sum_{i,j} |z_{ij}\boldsymbol{p}_j - \mathcal{M}_i \boldsymbol{P}_j|^2,$$

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$$E_j \stackrel{\text{def}}{=} \sum_{i=1}^{m} |z_{ij}\boldsymbol{p}_j - \mathcal{M}_i \boldsymbol{P}_j|^2.$$

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which must satisfy

$$0 = \frac{\partial E_j}{\partial \boldsymbol{P}_j} = 2 \sum_{i=1}^{m} \mathcal{M}_i^T (z_{ij} p_{ij} - \mathcal{M}_i \boldsymbol{P}_j),$$

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$$\mathcal{M}^T d_j = \mathcal{M}^T \mathcal{M} \boldsymbol{P}_j \Longleftrightarrow \boldsymbol{P}_j = \mathcal{M}^\dagger d_j,$$

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so

$$\mathcal{M}^T \boldsymbol{d}_j = \mathcal{M}^T \mathcal{M} \boldsymbol{P}_j \Longleftrightarrow \boldsymbol{P}_j = \mathcal{M}^\dagger \boldsymbol{d}_j,$$

and thus

$$E_j = |(\mathrm{Id} - \mathcal{M}\mathcal{M}^\dagger)\boldsymbol{d}_j|^2.$$

Find z_{ij} that best fit $\mathcal{P}$ and $\mathcal{M}$…

Take SVD of M

$$\mathcal{M}=\mathcal{U}\mathcal{W}\mathcal{V}^{T}$$

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and rewrite

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as

$$E_j = |[\mathrm{Id} - \mathcal{U}\mathcal{U}^T]d_j|^2 = 1 - |\mathcal{U}d_j|^2.$$

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We need to thus maximize $|\mathcal{U}\mathbf{d}_j|$ under constraint $|\mathbf{d}_j|^2=1$

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which is the same as maximizing $|\mathcal{R}_j z_j|^2$ with $|\mathcal{Q}_j z_j|^2 = 1$ where $\mathcal{R}_j \stackrel{\text{def}}{=} \mathcal{U}^T \mathcal{Q}_j$

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Finally this can be expressed as a solution to a generalized eigenvalue problem to find $\mathbf{z}_j$ corresponding to largest λ s.t. $\mathcal{R}_j^T \mathcal{R}_j z_j = \lambda \mathcal{Q}_j^T \mathcal{Q}_j z_j$

with

$$d_j = \mathcal{Q}_j z_j, \quad \text{where} \quad \mathcal{Q}_j \stackrel{\text{def}}{=} \begin{pmatrix} p_{1j} & 0 & \dots & 0 \\ 0 & p_{2j} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & p_{mj} \end{pmatrix},$$

Iterative approach to Projective Factorization

$$\mathcal{D} = \mathcal{M}\mathcal{P}, \qquad \mathcal{D} \stackrel{\text{def}}{=} \begin{pmatrix} z_{11}\boldsymbol{p}_{11} & z_{12}\boldsymbol{p}_{12} & \dots & z_{1n}\boldsymbol{p}_{1n} \\ z_{21}\boldsymbol{p}_{21} & z_{22}\boldsymbol{p}_{22} & \dots & z_{2n}\boldsymbol{p}_{2n} \\ \dots & \dots & \dots & \dots \\ z_{m1}\boldsymbol{p}_{m1} & z_{m2}\boldsymbol{p}_{m2} & \dots & z_{mn}\boldsymbol{p}_{mn} \end{pmatrix}, \; \mathcal{M} \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \dots \\ \mathcal{M}_m \end{pmatrix} \text{ and } \mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2 & \dots & \boldsymbol{P}_n \end{pmatrix}.$$

– Assume projective depths z_{ij} are known, and compute $\mathcal{M}$ and $\mathcal{P}$, using singular value decomposition to factor $\mathcal{D}$.

– Assume $\mathcal{M}$ and $\mathcal{P}$ ⊣∇⌉ \≀⊒ ⌊≀∇∇⌉⌋⊔ ⊣\⌈ solve directly for the values of the projective depths.

Find z_{ij} that best fit $\mathcal{P}$ and $\mathcal{M}$…

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Iterative approach to Projective Factorization

$$\mathcal{D} = \mathcal{M}\mathcal{P}, \qquad \mathcal{D} \stackrel{\text{def}}{=} \begin{pmatrix} z_{11}\boldsymbol{p}_{11} & z_{12}\boldsymbol{p}_{12} & \dots & z_{1n}\boldsymbol{p}_{1n} \\ z_{21}\boldsymbol{p}_{21} & z_{22}\boldsymbol{p}_{22} & \dots & z_{2n}\boldsymbol{p}_{2n} \\ \dots & \dots & \dots & \dots \\ z_{m1}\boldsymbol{p}_{m1} & z_{m2}\boldsymbol{p}_{m2} & \dots & z_{mn}\boldsymbol{p}_{mn} \end{pmatrix}, \ \mathcal{M} \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \dots \\ \mathcal{M}_m \end{pmatrix} \text{ and } \mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2 & \dots & \boldsymbol{P}_n \end{pmatrix}.$$

– Assume projective depths z_{ij} are known, and compute $\mathcal{M}$ and $\mathcal{P}$, using singular value decomposition to factor $\mathcal{D}$.

…use Tomasi and Kanade SVD algorithm

– Assume $\mathcal{M}$ and $\mathcal{P}$ $\dashv\nabla\rceil \ \backslash\wr\sqsupseteq \ \rfloor\wr\nabla\nabla\rceil\rfloor\sqcup \ \dashv\backslash\lceil$ solve directly for the values of the projective depths.

Find z_{ij} that best fit $\mathcal{P}$ and $\mathcal{M}$…

… express as solution to generalized eigenvalue problem.

Proj. SFM algorithm

1. Compute an initial estimate of the projective depths z_{ij}, with $i = 1, \ldots, m$ and $j = 1, \ldots, n$.
2. Normalize each column of the data matrix $\mathcal{D}$.
3. Repeat:

   (a) use singular value decomposition to compute the $2m \times 4$ matrix $\mathcal{M}$ and the $4 \times n$ matrix $\mathcal{P}$ that minimize $|\mathcal{D} - \mathcal{MP}|^2$;

   (b) for $j = 1$ to n, compute the matrices $\mathcal{R}_j$ and $\mathcal{Q}_j$ and find the value of z_j that maximize $|\mathcal{R}_j z_j|^2$ under the constraint $|\mathcal{Q}_j z_j|^2 = 1$ as the solution of a generalized eigenvalue problem;

   (c) update the value of $\mathcal{D}$ accordingly;

   until convergence.

(a)

(b)

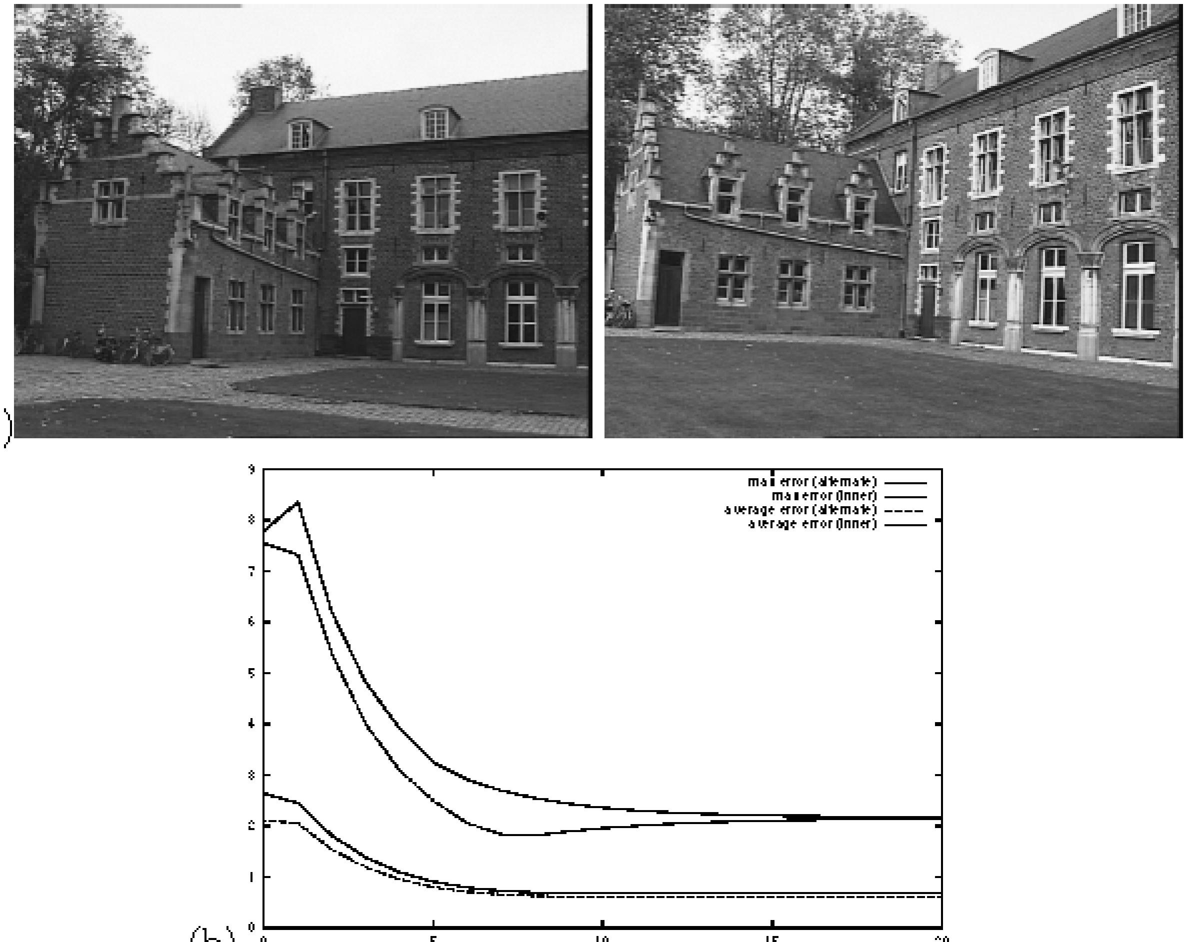

FIGURE 14.5: Iterative projective estimation of camera motion and scene structure: (a) the first and last images in the sequence; (b) plot of the average and maximum reprojection error as a function of iteration number. Two experiments were conducted: in the first one (alternate) alternate images in the sequence are used as training and testing datasets; in the second experiment (inner), the first five and last five pictures were used as training set, and the remaining images were used for testing. In both cases, the average error falls below 1 pixel after 15 iterations. Reprinted from [Mahamud and Hebert, 2000], Figure 4.

Bundle adjustment

Given initial estimates for the matrices $\mathcal{M}$i ($i = 1, \ldots, m$) and vectors Pj ($j = 1, \ldots, n$), we can refine these estimates by using non-linear least squares to minimize the global error measure

$$E = \frac{1}{mn} \sum_{i,j} \left[\left(u_{ij} - \frac{\boldsymbol{m}_{i1} \cdot \boldsymbol{P}_j}{\boldsymbol{m}_{i3} \cdot \boldsymbol{P}_j} \right)^2 + \left(v_{ij} - \frac{\boldsymbol{m}_{i2} \cdot \boldsymbol{P}_j}{\boldsymbol{m}_{i3} \cdot \boldsymbol{P}_j} \right)^2 \right].$$

Euclidean upgrade

Given a camera with known intrinsic parameters, we can take the calibration matrix to be the identity and write the perspective projection equation in some Euclidean world coordinate system as

$$\boldsymbol{p} = \frac{1}{z}\begin{pmatrix}\mathcal{R} & \boldsymbol{t}\end{pmatrix}\begin{pmatrix}\boldsymbol{P} \\ 1\end{pmatrix} = \frac{1}{\lambda z}\begin{pmatrix}\mathcal{R} & \lambda\boldsymbol{t}\end{pmatrix}\begin{pmatrix}\lambda\boldsymbol{P} \\ 1\end{pmatrix}$$

for any non-zero scale factor λ.

If $\mathcal{M}_i$ and $\mathbf{P}_j$ denote the shape and motion parameters measured in some Euclidean coordinate system, there must exist a 4 ×4 matrix $\mathcal{Q}$ such that

$$\hat{\mathcal{M}}_i = \mathcal{M}_i\mathcal{Q} \text{ and } \hat{\boldsymbol{P}}_j = \mathcal{Q}^{-1}\boldsymbol{P}_j.$$

Euclidean upgrade

$$\hat{\mathcal{M}}_i = \rho_i \mathcal{K}_i \begin{pmatrix} \mathcal{R}_i & \boldsymbol{t}_i \end{pmatrix},$$

where ρ_i accounts for the unknown scale of $\mathcal{M}_i$, and $\mathcal{K}_i$ is a calibration matrix

$$\mathcal{M}_i \mathcal{Q}_3 = \rho_i \mathcal{K}_i \mathcal{R}_i.$$

the 3×3 matrices $\mathcal{M}_i \mathcal{Q}_3$ are in this case scaled rotation matrices.

$$\begin{aligned}
\boldsymbol{m}_{i1}^T \mathcal{Q}_3 \mathcal{Q}_3^T \boldsymbol{m}_{i2} &= 0, \\
\boldsymbol{m}_{i2}^T \mathcal{Q}_3 \mathcal{Q}_3^T \boldsymbol{m}_{i3} &= 0, \\
\boldsymbol{m}_{i3}^T \mathcal{Q}_3 \mathcal{Q}_3^T \boldsymbol{m}_{i1} &= 0, \\
\boldsymbol{m}_{i1}^T \mathcal{Q}_3 \mathcal{Q}_3^T \boldsymbol{m}_{i1} - \boldsymbol{m}_{i2}^T \mathcal{Q}_3 \mathcal{Q}_3^T \boldsymbol{m}_{i2} &= 0, \\
\boldsymbol{m}_{i2}^T \mathcal{Q}_3 \mathcal{Q}_3^T \boldsymbol{m}_{i2} - \boldsymbol{m}_{i3}^T \mathcal{Q}_3 \mathcal{Q}_3^T \boldsymbol{m}_{i3} &= 0.
\end{aligned}$$

Euclidean upgrade

- can also perform upgrade when partial calibration is known
- even just knowledge of zero-skew

Euclidean upgrade

FIGURE 14.6: A synthetic texture-mapped image of a castle constructed via projective motion analysis followed by a Euclidean upgrade. The principal point is assumed to be known. Reprinted from [Pollefeys, 1999], Figure 6.13.

Administrivia

Pset 4 delayed one day; on web tomorrow.

Last lecture readings: 22

Today's F&P readings: 13.0, 13.1, 13.4, 13.5

Today:

- Projective spaces
- Cross ratio
- Geometric reconstruction
- Factorization algorithm
- Euclidean upgrade

[Most figures from F&P]