6.825 Recitation Problems: Lec. 3-5

Solutions

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1 Lecture 3: Logic

Which of these are legal sentences? Give fully parenthesized expressions.

- $(P \Rightarrow Q) \Rightarrow R$
- $P, R \Rightarrow Q$ **Not legal!**
- $(A \wedge (B \vee C \vee (\neg D))) \Leftrightarrow (\neg(\neg Z))$
- $P(Q)$ **Not legal in propositional logic**

2 Lecture 4: Conjunctive Normal Form

2.1 Converting to CNF

1. $(A \rightarrow B) \rightarrow C$
   $\neg(\neg A \vee B) \vee C$
   $(A \wedge \neg B) \vee C$
   $(A \vee C) \wedge (\neg B \vee C)$
2. $A \rightarrow (B \rightarrow C)$
   $\neg A \vee \neg B \vee C$
3. $(A \rightarrow B) \vee (B \rightarrow A)$
   $(\neg A \vee B) \vee (\neg B \vee A)$
   True
4. $(\neg P \rightarrow (P \rightarrow Q))$
   $\neg\neg P \vee (\neg P \vee Q)$
   $P \vee \neg P \vee Q$
   True
5. $(P \rightarrow (Q \rightarrow R)) \rightarrow (P \rightarrow (R \rightarrow Q))$
   $\neg(\neg P \vee \neg Q \vee R) \vee (\neg P \vee \neg R \vee Q)$
   $(P \wedge Q \wedge \neg R) \vee (\neg P \vee \neg R \vee Q)$
   $(P \vee \neg P \vee \neg R \vee Q) \wedge (Q \vee \neg P \vee \neg R \vee Q) \wedge (\neg R \vee \neg P \vee \neg R \vee Q)$
   $\neg P \vee Q \vee \neg R$

6. $(P \to Q) \to ((Q \to R) \to (P \to R))$

   $\neg(\neg P \lor Q) \lor (\neg(\neg Q \lor R) \lor (\neg P \lor R))$

   $(P \land \neg Q) \lor ((Q \land \neg R) \lor (\neg P \lor R))$

   $(P \land \neg Q) \lor ((Q \lor \neg P \lor R) \land (\neg R \lor \neg P \lor R))$

   $(P \land \neg Q) \lor Q \lor \neg P \lor R$

   $(P \lor \neg P \lor Q \lor R) \land (\neg P \lor Q \lor \neg Q \lor R)$

   True

2.2 DPLL

1. How would you modify DPLL so that it

   - returns a satisfying assignment if there is one, and false otherwise

     Every time you assign a value to a variable, keep track of that assignment, and return the assignments when returning *true*.

   - returns *all* satisfying assignments

     (a) Remove the pure literal rule

     (b) When DPLL($\phi(v)$), store the solution and go onto DPLL($\phi(\neg v)$)

2. Would using DPLL to return all satisfying assignments be any more efficient that simply listing all the assignments and checking to see whether they're satisfying? Why or why not?

   Using DPLL would be more efficient because DPLL applies unit clause rules to make forced choices, and then simplifies the sentence to prune out the search space.

3 Lecture 5: FOL

3.1 Recitation Problems I

For each of the following sentences, determine whether it is true or false in the interpretation I we've been using:

1. $\forall x.above(x, Fred)$

   False

2. $\forall x.above(x, Hat(x))$

   Undefined

3. $\forall x.oval(x) \to \exists y.above(y, x)$

   False

4. $square(Hat(Hat(Fred)))$

   Undefined

5. $\forall x.above(x, Fred) \to square(x)$

   False

6. $\exists x.\forall y.circle(y) \to above(y, x)$

   True

3.2 Recitation Problems II

1. Somebody loves Jane
   $\exists x.loves(x, Jane)$
2. For every mountain in England, there is a higher mountain in Scotland
   $\forall x.(mountain(x) \wedge inEngland(x)) \rightarrow \exists y.(mountain(y) \wedge inScotland(y) \wedge higher(y, x))$
3. There are at least two mountains in England
   $\exists x, y.mountain(x) \wedge moutain(y) \wedge inEngland(x) \wedge inEngland(y) \wedge x \neq y$
4. You can fool all of the people some of the time
   $\forall p.\exists t.canfool(p, t)$
   or
   $\exists t.\forall p.canfool(p, t)$
5. The Barber of Seville shaves all men who do not shave themselves
   $\forall x.\neg shaves(x, x) \rightarrow shaves(BofS, x)$
6. The only good extraterrestrial is a dead extraterrestrial
   $\forall x.ET(x) \wedge good(x) \rightarrow dead(x)$
7. There is exactly one coin in the box
   $\exists x.coin(x) \wedge inbox(x) \wedge \forall y.(coin(y) \wedge inbox(y) \rightarrow x = y)$
8. No mountain is higher than itself
   $\forall x.mountain(x) \rightarrow \neg higher(x, x)$
   or
   $\neg \exists x.mountain(x) \wedge higher(x, x)$
9. All students get good grades if they study
   $\forall x.student(x) \wedge study(x) \rightarrow getGoodGrades(x)$
10. Some students get good grades if they study
    $\exists x.student(x) \wedge study(x) \rightarrow getGoodGrades(x)$

3.3 Recitation Problems III

1. $\forall x.h(x) \rightarrow g(x)$
   $\forall x.f(x) \rightarrow g(x)$
   $\exists x.f(x) \wedge h(x)$
   An interpretation that makes the first two sentences true and the third sentence false:
   $U = \{A, B\}$
   $I(f) = \{A\}$
   $I(g) = \{A, B\}$
   $I(h) = \{B\}$
2. $\forall x.\exists y.f(x, y)$
   $\exists y.\forall x.f(x, y)$
   An interpreation that makes that first sentence true and the second sentence false:
   $U = \{A, B, C\}$
   $I(f) = \{< A, B >, < B, C >, < C, A >\}$

3. $\forall x.(f(x) \rightarrow g(A))$

   $(\forall x.f(x)) \rightarrow g(A)$

   There is no interpretation that makes the first sentence true and the second sentence false.

   Reason: For the second sentence to be false, $\forall x.f(x)$ has to be *true*, **and** $g(A)$ has to be *false*. With these two requirements, we can see that the first sentence cannot be true because $f(x)$ is true for $\forall x$, and $g(A)$ is false.

   However, if we replace $\forall x$ with $\exists x$,

   $\exists x.(f(x) \rightarrow g(A))$

   $(\exists x.f(x)) \rightarrow g(A)$

   Then the following interpretation makes the first sentence true and the second sentence false.

   $U = \{A, B\}$

   $f = \{B\}$

   $g = \{B\}$

3.4 Recitation Problems IV

For each group of sentences, give an interpretation in which all sentences are true.

1. $(\forall x.p(x) \vee q(x)) \rightarrow \exists x.r(x)$

   $\forall x.r(x) \rightarrow q(x)$

   $\exists x.p(x) \wedge \neg q(x)$

   Interpretation:

   $U = \{A, B\}$

   $I(p) = \{A\}$

   $I(q) = \{B\}$

   $I(r) = \{B\}$

2. $\forall x.\neg f(x, x)$

   $\forall x, y, z.f(x, y) \wedge f(y, z) \rightarrow f(x, z)$

   $\forall x.\exists y.f(x, y)$

   There is no interpretation in a finite universe that makes all of these sentences true. However, if you consider an infinite universe, (e.g., real numbers) and a *greater than* function ($>$), these sentences are all true.

   Interpretation:

   $U = \mathbb{R}$

   $I(f) = >$

3. $\forall x.\exists y.f(x, y)$

   $\forall x.(g(x) \rightarrow \exists y.f(y, x))$

   $\exists x.g(x)$

   $\forall x.\neg f(x, x)$

   Interpretation:

   $U = \{A, B\}$

   $I(f) = \{< A, B >, < B, A >\}$

   $I(g) = \{A\}$