6.825 Techniques in Artificial Intelligence

First-Order Logic

- Propositional logic only deals with "facts", statements that may or may not be true of the world, e.g. "It is raining". But, one cannot have variables that stand for books or tables.
- In first-order logic variables refer to things in the world and, furthermore, you can quantify over them – to talk about all of them or some of them without having to name them explicitly.

FOL motivation

- Statements that cannot be made in propositional logic but can be made in FOL.
  - When you paint a block with green paint, it becomes green.
    - In propositional logic, one would need a statement about every single block, one cannot make the general statement about all blocks.
  - When you sterilize a jar, all the bacteria are dead.
    - In FOL, we can talk about all the bacteria without naming them explicitly.
  - A person is allowed access to this Web site if they have been formally authorized or they are known to someone who has access.

FOL syntax

- Term
  - Constant symbols: Fred, Japan, Bacterium39
  - Variables: x, y, a
  - Function symbol applied to one or more terms: F(x), F(F(x)), Mother-of(John)
- Sentence
  - A predicate symbol applied to zero or more terms: on(a,b), sister(Jane, Joan), sister(Mother-of(John), Jane), its-raining()
  - $t_1=t_2$
  - For v a variable and Φ a sentence, then ∀v.Φ and ∃v.Φ are sentences.
  - Closure under sentential operators: ∧ v → ¬ ( )

FOL Interpretations

- Interpretation I
  - U set of objects; domain of discourse; universe
  - Maps constant symbols to elements of U
  - Maps predicate symbols to relations on U (binary relation is a set of pairs)
  - Maps function symbols to functions on U

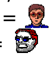
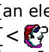
Basic FOL Semantics

Denotation of terms (naming)

- I(Fred) if Fred is constant, then given
- I(x) undefined
- I(F(term)) I(F)(I(term))

$\models_I P(t_1, \ldots, t_n)$ iff $<I(t_1), \ldots, I(t_n)> \in I(P)$

brother(John, Joe)??

- I(John) =  [an element of U]
- I(Joe) =  [an element of U]
- I(brother) = {<,>,<,>, <…,…>, … }
- $\models_I$ brother(John, Joe)

Semantics of Quantifiers

Extend an interpretation I to bind variable x to element a ∈ U: $I_{x/a}$

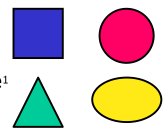
- $\models_I \forall x.\Phi$ iff $\models_{Ix/a} \Phi$ for all a ∈ U
- $\models_I \exists x.\Phi$ iff $\models_{Ix/a} \Phi$ for some a ∈ U

- Quantifier applies to formula to right until an enclosing right parenthesis:

$$(\quad x.p(x) \quad q(x)) \qquad x.r(x) \qquad q(x)$$

FOL Example Domain

- U = {■,▲,●,⬬}
- Constants: Fred
- Preds: above², circle¹, oval¹, square¹
- Function: Hat
- I(Fred) = ▲
- I(above) = {<■,▲>,<●,⬬>}
- I(circle) = {<●>}
- I(oval) = {<●>,<⬬>}
- I(Hat) = {<▲,■>,<⬬,●>}
- I(square) = {<▲>}



The Real World

FOL Example

- I(Fred) = ▲
- I(above) = {<■,▲>,<●,⬬>}
- I(circle) = {<●>}
- I(oval) = {<●>,<⬬>}
- I(Hat) = {<▲,■>,<⬬,●>}
- I(square) = {<▲>}

- $\models_I$ square(Fred)?
- $\not\models_I$ above(Fred, Hat(Fred))?
  - I(Hat(Fred)) = ■
  - $\not\models_I$ above(▲, ■) ?
- $\models_I$ ∃x. oval(x)?
  - $\models_{Ix/\bullet}$ oval(x)

FOL Example: Continued

- I(Fred) = ▲
- I(above) = {<■,▲>,<●,⬬>}
- I(circle) = {<●>}
- I(oval) = {<●>,<⬬>}
- I(Hat) = {<▲,■>,<⬬,●>}
- I(square) = {<▲>}

- $\models_I$ ∀x. ∃y. above(x,y) v above(y, x)
  - $\models_{Ix/\blacktriangle}$ ∃y. …
  - $\models_{Ix/\blacktriangle,\ y/\blacksquare}$ above(x,y) v above(y,x)
- $\not\models_I$ ∀ x. ∀ y. above(x,y) v above(y,x)
  - $\not\models_{Ix/\blacksquare,\ y/\bullet}$ above(x,y) v above(y,x)

Recitation Problems: I

For each of the following sentences, determine whether it is true or false in the interpretation I we've been using:

- 1. x.above(x,Fred)
- 2. x.above(x,Hat(x))
- 3. x.oval(x) y.above(y,x)
- 4. square(Hat(Hat(Fred)))
- 5. x.above(x,Fred)   square(x)
- 6. x. y.circle(y)   above(y,x)

Writing FOL

- Cats are mammals [cat¹, mammal¹]
  - ∀ x. cat(x) → mammal(x)
- Jane is a tall surveyor [tall¹, surveyor¹, Jane]
  - tall(Jane) ∧ surveyor(Jane)
- A nephew is a sibling's son [nephew², sibling², son²]
  - ∀xy. [nephew(x,y) ↔ ∃z . [sibling(y,z) ∧ son(x,z)]]
- A maternal grandmother … [functions: mgm, mother-of]
  - ∀xy. x=mgm(y) ↔ ∃z. x=mother-of(z) ∧ z=mother-of(y)
- Everybody loves somebody [loves²]
  - ∀x. ∃y. loves(x,y)
  - ∃y. ∀x. loves(x,y)

Writing More FOL

- Nobody loves Jane
  - ∀x. ¬ loves(x,Jane)
  - ¬∃x. loves(x,Jane)
- Everybody has a father
  - ∀ x. ∃ y. father(y,x)
- Everybody has a father and a mother
  - ∀ x. ∃ yz. father(y,x) ∧ mother(z,x)
- Whoever has a father, has a mother
  - ∀ x.[[∃ y. father(y,x)] → [∃ y. mother(y,x)]]

Writing More FOL

- Nobody loves Jane
  - $\forall x.\ \neg$ loves(x,Jane)
  - $\neg\exists x.$ loves(x,Jane)
- Everybody has a father
  - $\forall$ x. $\exists$ y. father(y,x)
- Everybody has a father and a mother
  - $\forall$ x. $\exists$ yz. father(y,x) $\wedge$ mother(z,x)
- Whoever has a father, has a mother
  - $\forall$ x.[[$\exists$ y. father(y,x)] $\rightarrow$ [$\exists$ y. mother(y,x)]]

Recitation Problems: II

For each of the following English sentences, write a corresponding sentence in FOL

1. Somebody loves Jane.
2.
3.
4.
5.
6.