6.825 Techniques in Artificial Intelligence

Resolution Theorem Proving: First Order Logic

Resolution with variables

Clausal form

Resolution with Variables

$\alpha \vee \phi$ [rename]

$\neg\psi \vee \beta$ [rename]   MGU(ϕ,ψ) = θ

$(\alpha \vee \beta)\theta$

P(x) v Q(x,y)

¬ P(A) v R(B,z)

(Q(x,y) v R(B,z))θ

Q(A,y) v R(B,z)

θ = {x/A}

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∀ x.   ¬ P(A) v R(B,x)

Scope of var is local to a clause.
Use renaming to keep vars distinct

All vars implicitly
univ. quantified

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∀ x_1y.   P(x_1) v Q(x_1,y)

∀ x_2.   ¬ P(A) v R(B,x_2)

θ = {x/A}

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∀ x_2.   ¬ P(A) v R(B,x_2)

(Q(x_1,y) v R(B,x_2))θ

Q(A,y) v R(B,x_2)

θ = {x/A}

θ = {x_1/A}

Resolution

Input are sentences in conjunctive normal form with no apparent quantifiers (implicit universal quantifiers).

How do we go from the full range of sentences in FOL, with the full range of quantifiers, to sentences that enable us to use resolution as our single inference rule?

We will convert the input sentences into a new normal form called clausal form.

Converting to Clausal Form

1. Eliminate →, ↔

$\alpha \rightarrow \beta$	$\neg\alpha \vee \beta$

2. Drive in ¬

$\neg(\alpha \vee \beta)$	$\neg\alpha \wedge \neg\beta$
$\neg(\alpha \wedge \beta)$	$\neg\alpha \vee \neg\beta$
$\neg\neg\alpha$	α
$\neg\exists x.\ P(x)$	$\forall x.\ \neg P(x)$
$\neg\forall x.\ P(x)$	$\exists x.\ \neg P(x)$

3. Rename variables apart

$$\forall x.\ \exists y.\ (P(x) \rightarrow \forall x.\ Q(x,y))$$
$$\forall x_1.\ \exists y_2.\ (P(x_1) \rightarrow \forall x_3.Q(x_3,y_2))$$

Converting to Clausal Form, II

4. Skolemize
   - Substitute brand new name for each existentially quantified variable
   - Substitute a new function of all universally quantified variables in enclosing scopes for each existentially quantified variable.
     - $\exists x.\ P(x) \Rightarrow P(Fred)$
     - $\exists x.\ P(x,y) \Rightarrow P(X_{11}, Y_{13})$
     - $\exists x.\ P(x) \wedge Q(x) \Rightarrow P(Blue) \wedge Q(Blue)$
     - $\exists y.\ \forall x.\ Loves(x,y) \Rightarrow \forall x.\ Loves(x, Englebert)$
     - $\forall x.\ \exists y.\ Loves(x,y) \Rightarrow \forall x.\ Loves(x, Beloved(x))$

5. Drop universal quantifiers
6. Convert to CNF
7. Rename the variables in each clause
   – $\forall x.\ P(x) \wedge\ Q(x) \Rightarrow\ \ \forall y.\ P(y) \wedge\ \ \forall z.\ Q(z)$

Example: Converting to clausal form

a. John owns a dog
$\exists x.\ D(x) \wedge O(J,x)$
$D(Fido) \wedge O(J, Fido)$

b. Anyone who owns a dog is a lover-of-animals
$\forall x.\ (\exists y.\ D(y) \wedge O(x,y)) \rightarrow L(x)$
$\forall x.\ (\neg\exists y.\ (D(y) \wedge O(x,y)) \vee L(x)$
$\forall x.\ \forall y.\ \neg(D(y) \wedge O(x,y)) \vee L(x)$
$\forall x.\ \forall y.\ \neg D(y) \vee \neg O(x,y) \vee L(x)$
$\neg D(y) \vee \neg O(x,y) \vee L(x)$

c. Lovers-of-animals do not kill animals
$\forall x.\ L(x) \rightarrow (\forall y.\ A(y) \rightarrow \neg K(x,y))$
$\forall x.\ \neg L(x) \vee (\forall y.\ A(y) \rightarrow \neg K(x,y))$
$\forall x.\ \neg L(x) \vee (\forall y.\ \neg A(y) \vee \neg K(x,y))$
$\neg L(x) \vee \neg A(y) \vee \neg K(x,y)$

More converting to clausal form

d. Either Jack killed Tuna or curiosity killed Tuna
$K(J,T) \vee K(C,T)$

e. Tuna is a cat
$C(T)$

f. All cats are animals
$\neg C(x) \vee A(x)$

Curiosity Killed the Cat

1	D(Fido)	a
2	O(J,Fido)	a
3	¬ D(y) v ¬ O(x,y) v L(x)	b
4	¬ L(x) v ¬ A(y) v ¬ K(x,y)	c
5	K(J,T) v K(C,T)	d
6	C(T)	e
7	¬ C(x) v A(x)	f
8	¬ K(C,T)	Neg
9	K(J,T)	5,8
10	A(T)	6,7 {x/T}
11	¬ L(J) v ¬ A(T)	4,9 {x/J, y/T}
12	¬ L(J)	10,11
13	¬ D(y) v ¬ O(J,y)	3,12 {x/J}
14	¬ D(Fido)	13,2 {x/Fido}
15	•	14,1

Proving validity

How do we use resolution refutation to prove something is valid?

Normally, we prove a sentence is entailed by the set of axioms

Valid sentences are entailed by the empty set of sentences

- ϕ is valid
- $\{\ \} \vDash \phi$ [empty set of sentences entails ϕ]
- $\{\ \} \vdash \phi$ [empty set of sentences proves ϕ]

To prove validity by refutation, negate the sentence and try to derive contradiction.

Proving validity: example

Prove validity of:
∃ x. (P(x) → P(A)) ∧ (P(x) → P(B))

¬ ∃ x. (P(x) → P(A)) ∧ (P(x) → P(B))
¬ ∃ x. ((¬ P(x) v P(A)) ∧ (¬ P(x) v P(B))
∀ x. ¬((¬ P(x) v P(A)) ∧ (¬ P(x) v P(B))
∀ x. ¬(¬ P(x) v P(A)) v ¬(¬ P(x) v P(B))
∀ x. (P(x) ∧ ¬ P(A)) v (P(x) ∧ ¬ P(B))
(P(x) ∧ ¬ P(A)) v (P(x) ∧ ¬ P(B))
(P(x) v P(x)) ∧ (P(x) v ¬ P(B)) ∧ (¬ P(A) v P(x)) ∧ (¬ P(A) v ¬ P(B))

1	P(x)	
2	P(x) v ¬ P(B)	
3	¬ P(A) v P(x)	
4	¬ P(A) v ¬ P(B)	
5	¬ P(B)	1,4 {x/A}
6	•	1,5 {x/B}

Recitation Problems

In each group, derive the last sentence from the others using resolution refutation.

∀ xy.F(x,y)
∀ xy.F(y,x)

∃ x.F(x)
∃ y.F(y)

∀ x.F(x)→ (G(x) v H(x))
G(A)↔ (H(A)∧ ¬ G(A))
¬ F(A)

∀ xyz.F(x,y)∧ F(y,z)→ F(x,z)
¬ F(x,x)
∀ xy.F(x,y)→ ¬ F(y,x)

∀ x.F(x)∨ G(x)
∃ x.¬ G(x)
∀ x.H(x)→ ¬ F(x)
∃ x.¬ H(x)

∀ x.∃ y.L(x,y)
∀ xy.L(x,y)→ H(x)
∀ x.H(x)

A Silly Recitation Problem

Symbolize the following argument, and then derive the conclusion from the premises using resolution refutation.

- Nobody who really appreciates Beethoven fails to keep silence while the Moonlight sonata is being played.
- Guinea pigs are hopelessly ignorant of music.
- No one who is hopelessly ignorant of music ever keeps silence while the moonlight sonata is being played.
- Therefore, guinea pigs never really appreciate Beethoven.

(Taken from a book by Lewis Carroll, logician and author of *Alice in Wonderland*.)

Another, Sillier Problem

You don’t have to do this one. It’s just for fun. Same type as the previous one. Also from Lewis Carroll.

- The only animals in this house are cats
- Every animal that loves to gaze at the moon is suitable for a pet
- When I detest an animal, I avoid it
- No animals are carnivorous unless they prowl at night
- No cat fails to kill mice
- No animals ever like me, except those that are in this house
- Kangaroos are not suitable for pets
- None but carnivorous animals kill mice
- I detest animals that do not like me
- Animals that prowl at night always love to gaze at the moon
- Therefore, I always avoid a kangaroo