Machine learning: lecture 18

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Topics

- Hidden markov models
  - dynamic programming, examples
- Representation and graphical models
  - variables and states
  - graphical models

Dynamic programming: review

$$\begin{array}{ccccccccc} s_0 & \rightarrow & s_1 & \rightarrow & s_2 & \rightarrow \ldots \rightarrow & s_{n-1} & \rightarrow & s_n \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbf{x}_0 & & \mathbf{x}_1 & & \mathbf{x}_2 & & \mathbf{x}_{n-1} & & \mathbf{x}_n \end{array}$$

- Let $\{s_0^{(t,i)}, \ldots, s_t^{(t,i)} = i\}$ be the most likely state sequence given $\mathbf{x}_0, \ldots, \mathbf{x}_t$ that is forced to end up in $s_t = i$ at time t. Then

$$\delta_t(i) = P(\mathbf{x}_0, \ldots, \mathbf{x}_t, s_0^{(t,i)}, \ldots, s_t^{(t,i)})$$

- We can evaluate these probabilities recursively by replacing each "sum" with a "max" in the forward propagation:

$$\begin{aligned} \delta_0(i) &= P_0(i) P_o(\mathbf{x}_t | s_0 = i), \\ \delta_t(i) &= \max_j \left\{ \, \delta_{t-1}(j) P_1(s_t = i | s_{t-1} = j) \, \right\} \times P_o(\mathbf{x}_t | s_t = i) \end{aligned}$$

Dynamic programming: review

- We can recover the most likely hidden state sequence from $\{\delta_t(\cdot)\}$ by retrospectively examining the "max" choices made in evaluating these probabilities

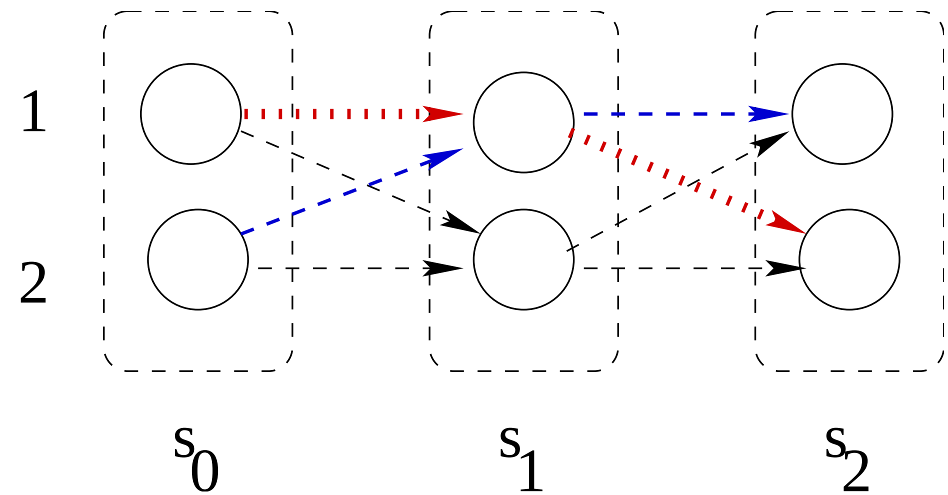
  We find the end state s_n^* of the most likely state sequence by maximizing over the probabilities associated with the most likely state sequences forced to land on different states at $t = n$:

$$s_n^* = \arg\max_j \delta_n(j)$$

  The recovery of the remaining states along the most likely path can be done recursively (backwards):

$$s_t^* = \arg\max_j \left\{ \delta_t(j) P_1(s_{t+1} = s_{t+1}^* | s_t = j) \right\}$$

Dynamic programming: review

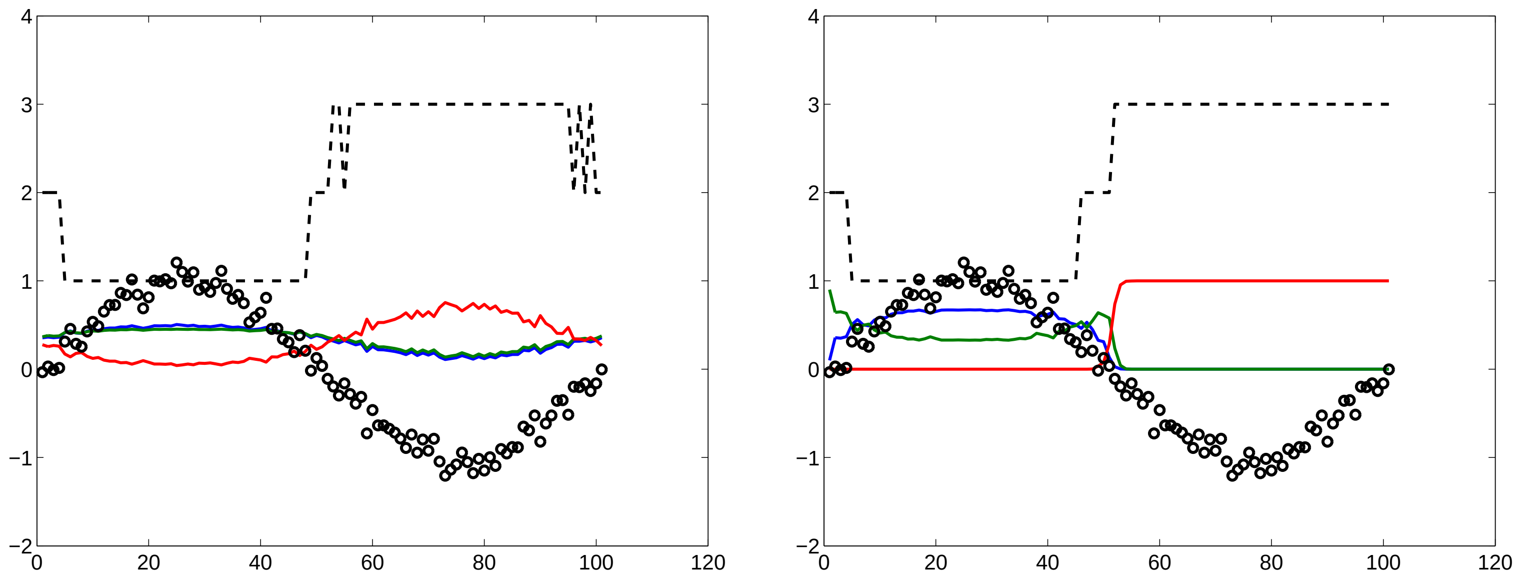


- The most likely path has the property that any partial path is also optimal:

  If $s_t^* = i$ then $\{s_0^*, \ldots, s_t^*\}$ is also the most likely state sequence forced to end up in $s_t = i$ at time t given only $\mathbf{x}_0, \ldots, \mathbf{x}_t$.

Dynamic programming: example

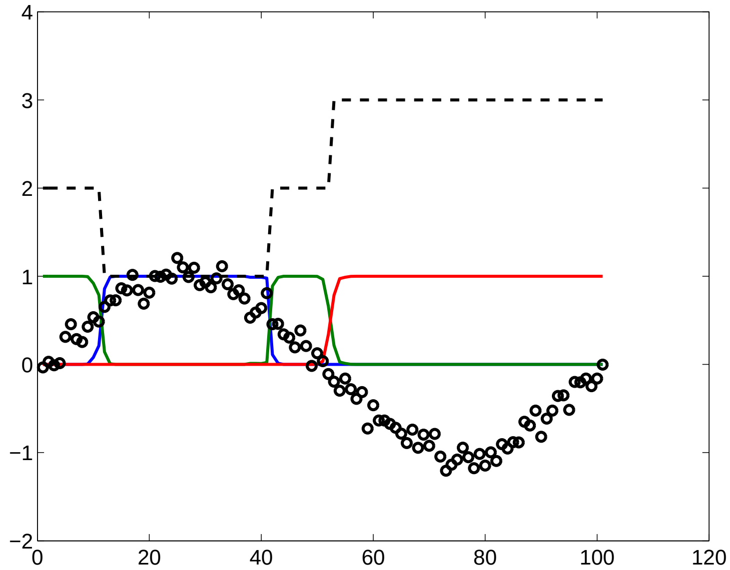
- Same example as in the EM case (3 states, Gaussian outputs)



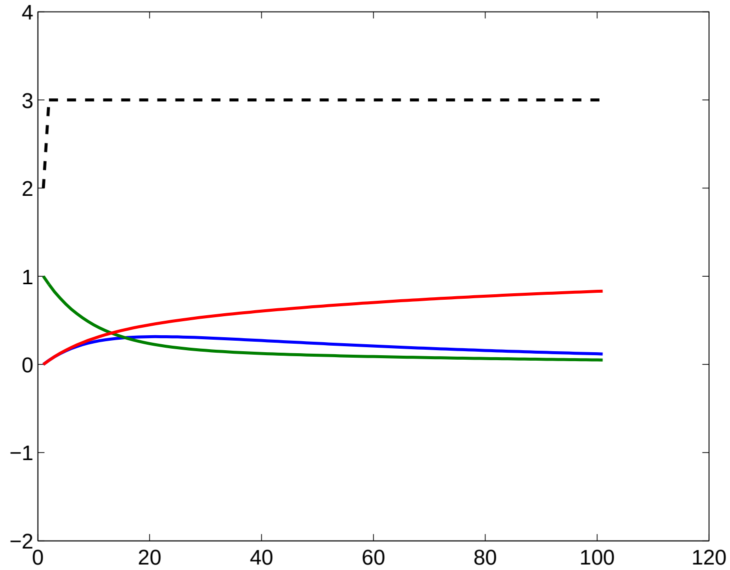
after 0 iterations   after 7 iterations

- The most likely hidden state sequence $\{s_0^*, \ldots, s_n^*\}$ need not agree with the most likely states derived from the posterior marginals $\gamma_t(i)$

Example cont'd



final

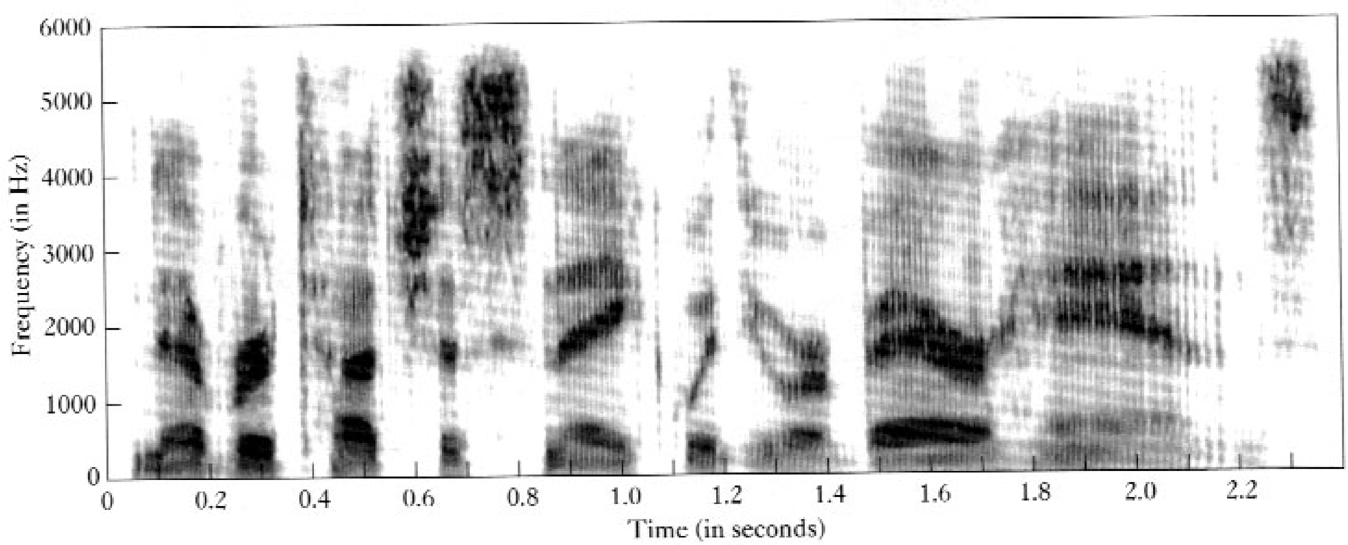


ML model, no observations

Examples: speech

- We can annotate or parse speech signals by evaluating the most likely hidden state sequence

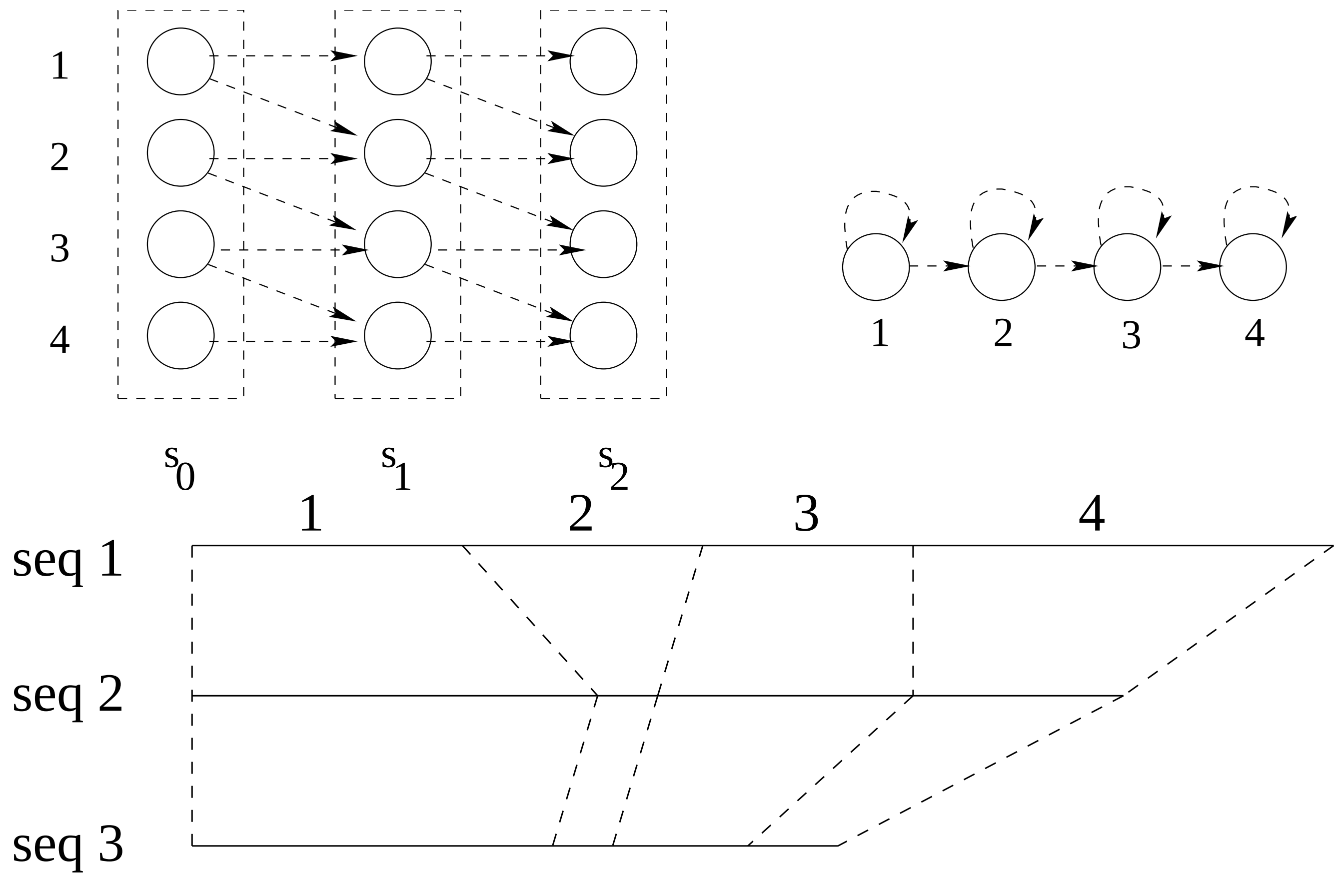
  A speech spectrogram example (refs)



Never touch a snake with your bare hands

Examples: alignment

- A "linear" HMM can be used to align sequences of observations



Topics

- Representation and graphical models
  - variables and states
  - graphical models

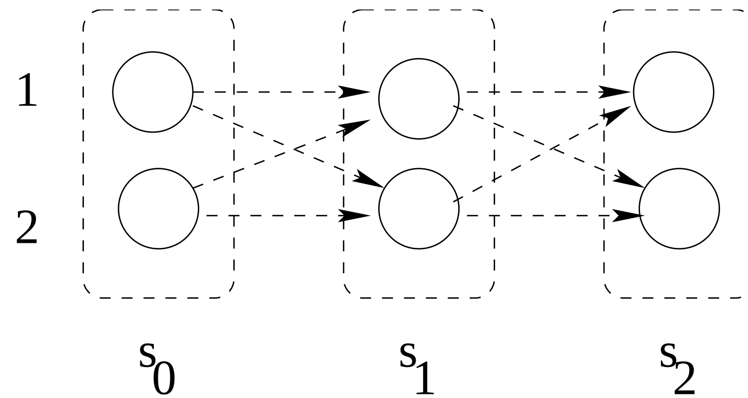
What is a good representation?

- Properties of good representations
  1. Explicit
  2. Compact
  3. Modular
  4. Permits efficient computation
  5. etc.

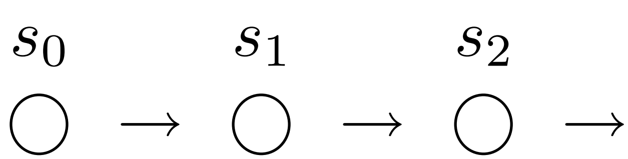
Representing the model structure

- Two possible representations of Markov models:

  1. in terms of state diagrams (nodes in the graph correspond to the possible values of the states)



  2. in terms of variables (nodes in the graph are variables):

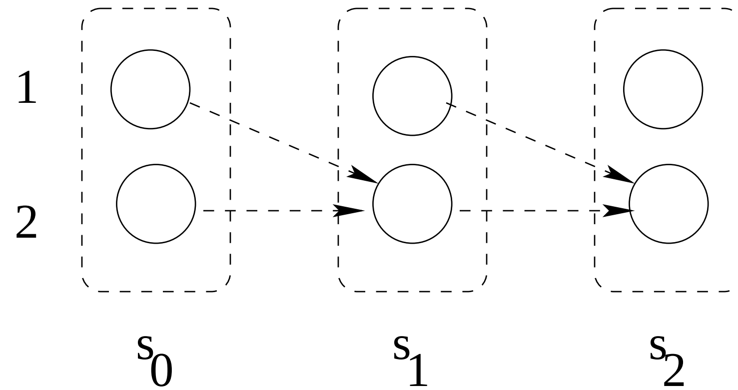


- The representations differ in terms of what aspects of the model are made *explicit*

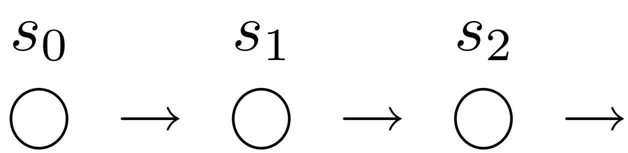
Model structure cont'd

- Case 1: *sparse transition* structure

1. State transition diagram is *explicit*



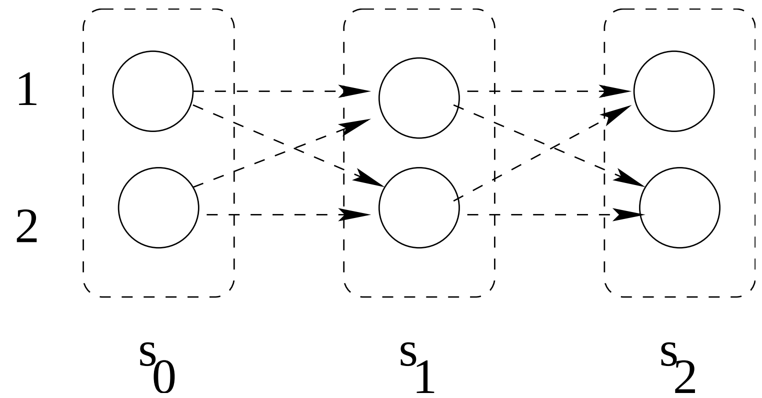
2. Representation in terms of variables leaves this *implicit*



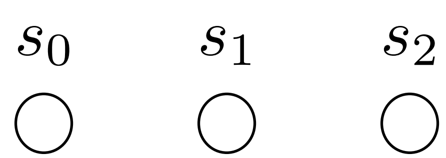
Model structure cont'd

- Case 2: successive states are *independent of each other*

1. State transition diagram is fully connected



2. Representation in terms of variables is *explicit*



Model structure cont'd

- Case 3: time series signals such as music may involve multiple relatively independent underlying processes operating at different *time scales*

  1. State transition diagram (argh #$& ...)

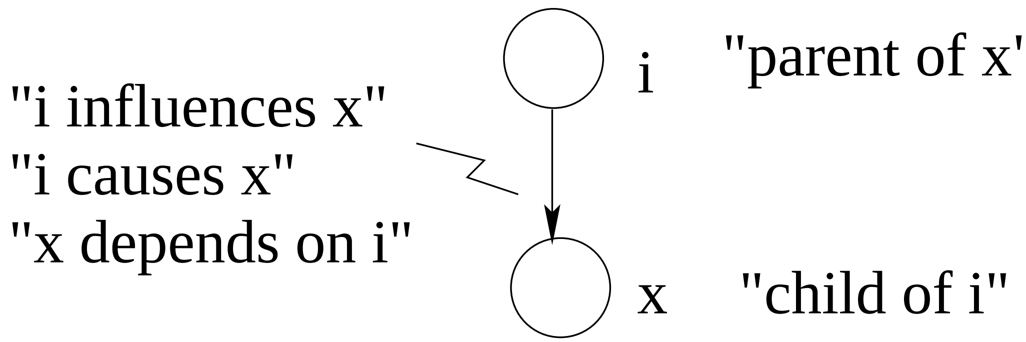
  2. In terms of variables (graph model)

$$
\begin{array}{cccccccccc}
s_0^{(1)} & & & & s_2^{(1)} & & & & s_4^{(1)} & \\
\bigcirc & & \longrightarrow & & \bigcirc & & \longrightarrow & & \bigcirc & \\
\downarrow & & & & \downarrow & & & & \downarrow & \\
\bigcirc & \longrightarrow & \bigcirc & \longrightarrow & \bigcirc & \longrightarrow & \bigcirc & \longrightarrow & \bigcirc & \longrightarrow \\
s_0^{(2)} & & s_1^{(2)} & & s_2^{(2)} & & s_3^{(2)} & & s_4^{(2)} &
\end{array}
$$

Graphical models

- Graph representions of probability models in terms of *variables* are known as *graphical models*

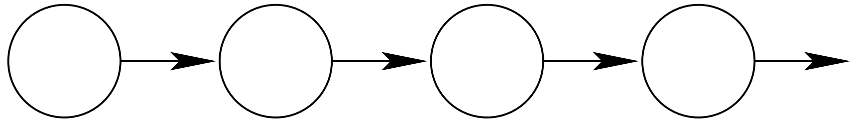
A mixture model as a graphical model



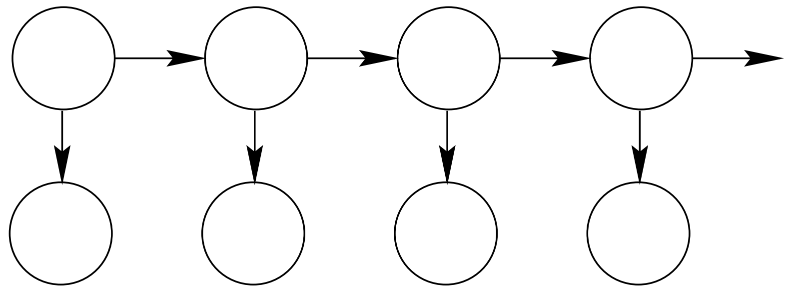
- Different types of graph models differ in terms of how we represent *dependencies* and *independencies* among the variables
  1. Bayesian networks (natural for "causal" relations)
  2. Markov random fields (natural for physical or symmetric relations)
  3. etc.

Bayesian networks: examples

A Markov chain:

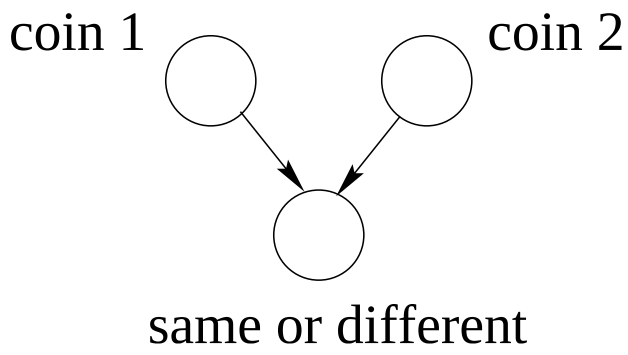


A hidden Markov model:



Qualitative inference

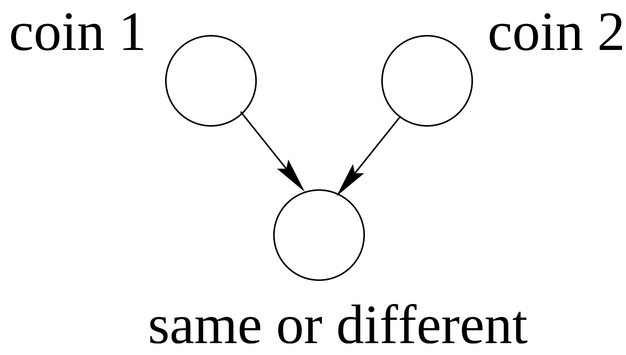
- The graph provides a qualitative description of the domain



$x_1 =$ 1st coin toss
$x_2 =$ 2nd coin toss
$x_3 =$ same?

Qualitative inference

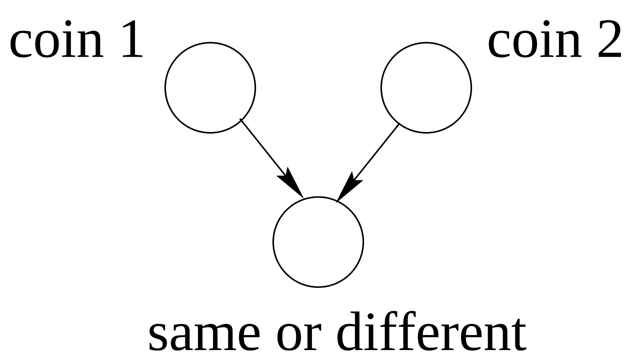
- The graph provides a qualitative description of the domain



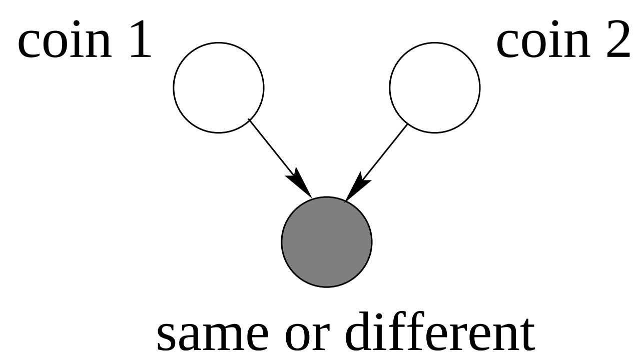
$x_1 =$ 1st coin toss

$x_2 =$ 2nd coin toss

$x_3 =$ same?



Marginal independence

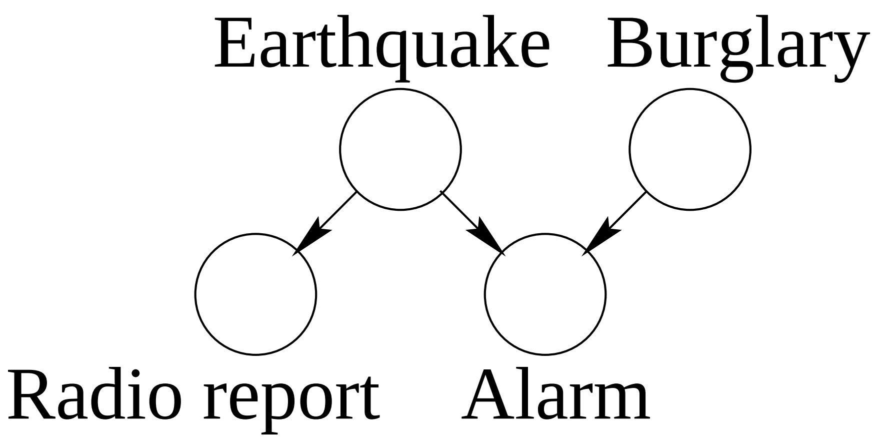


Induced dependence

Note that the induced dependence pertains to our beliefs about the outcomes of the coin tosses

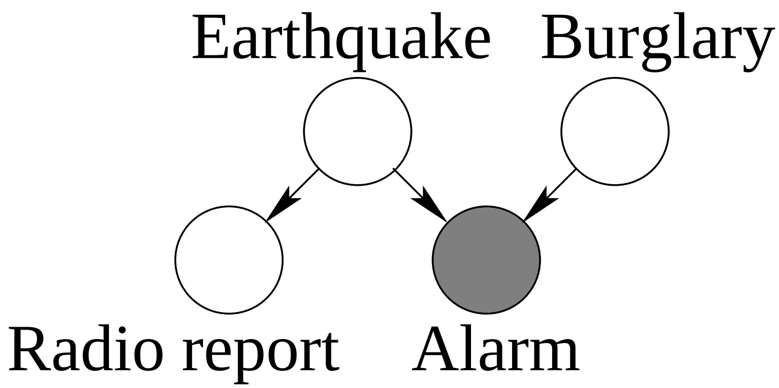
Qualitative inference cont'd

- Just by looking at the graph, we can determine what we can and cannot ignore (why important?)
  Marginal independence of "Earthquake" and "Burglary"

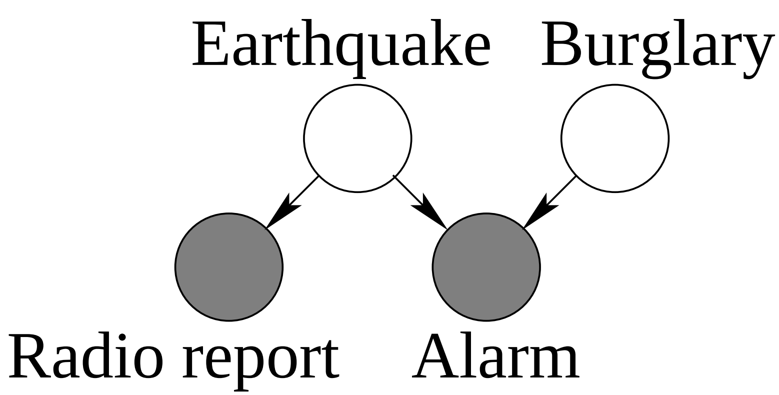


Qualitative inference cont'd

- Induced dependence:



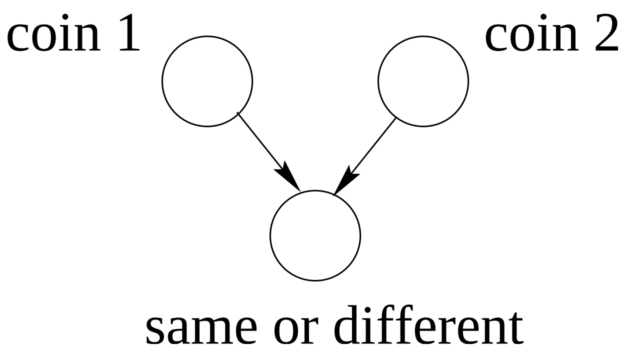
- Explaining away:



Two levels of description

- Graphical models need two levels of specification

1. Qualitative properties captured by a graph



x_1 = first coin toss

x_2 = second coin toss

x_3 = same?

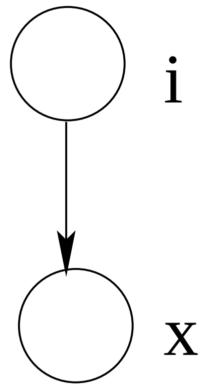
2. Quantitative properties specified by the associated probability distribution

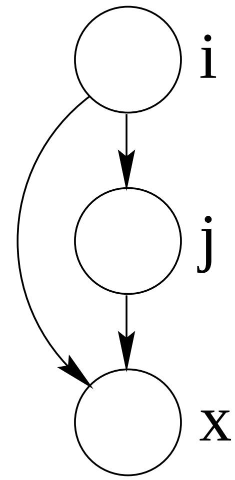
$$P(x_1, x_2, x_3) = P(x_1)\, P(x_2)\, P(x_3|x_1, x_2)$$

where, e.g.,

$$P(x_1 = heads) = 0.5$$
$$P(x_3 = same|x_1 = heads, x_2 = tails) = 0$$

More examples





Mixture model   hierarchical mixture model

- i and j correspond to the discrete choices in the mixture model
- $\mathbf{x}$ is the (vector) variable whose density we wish to model

- We cannot tell what the component distributions $P(\mathbf{x}|i)$ are based on the graph alone