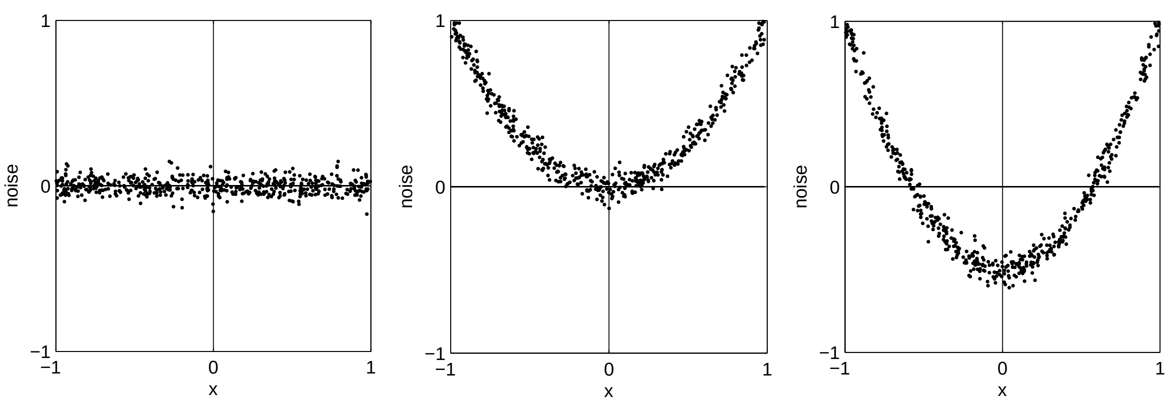
6.867 Machine learning

Mid-term exam

October 13, 2004

(2 points) Your name and MIT ID:

Problem 1



A **B** **C**

1. **(6 points)** Each plot above claims to represent prediction errors as a function of x for a trained regression model based on some dataset. Some of these plots could potentially be prediction errors for linear or quadratic regression models, while others couldn't. The regression models are trained with the least squares estimation criterion. Please indicate compatible models and plots.

	A	**B**	**C**
linear regression	( )	( )	( )
quadratic regression	( )	( )	( )

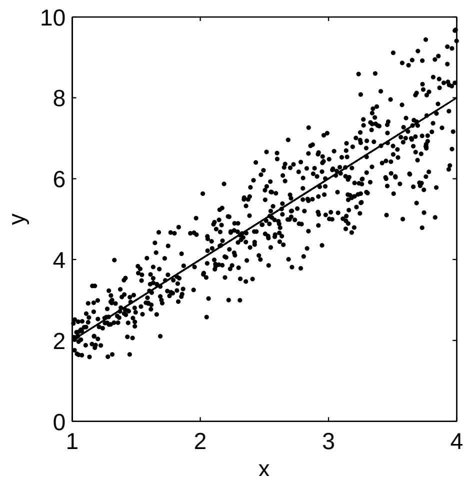
Problem 2

Here we explore a regression model where the noise variance is a function of the input (variance increases as a function of input). Specifically

$$y = wx + \epsilon$$

where the noise ϵ is normally distributed with mean 0 and standard deviation σx. The value of σ is assumed known and the input x is restricted to the interval $[1, 4]$. We can write the model more compactly as $y \sim N(wx, \sigma^2 x^2)$.

If we let x vary within $[1, 4]$ and sample outputs y from this model with some w, the regression plot might look like



1. **(2 points)** How is the ratio y/x distributed for a fixed (constant) x?

2. Suppose we now have n training points and targets $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$, where each x_i is chosen at random from $[1, 4]$ and the corresponding y_i is subsequently sampled from $y_i \sim N(w^* x_i, \sigma^2 x_i^2)$ with some true underlying parameter value w^*; the value of σ^2 is the same as in our model.

(a) **(3 points)** What is the maximum-likelihood estimate of w as a function of the training data?

(b) **(3 points)** What is the variance of this estimator due to the noise in the target outputs as a function of n and σ^2 for fixed inputs $x_1, \ldots, x_n$? For later utility (if you omit this answer) you can denote the answer as $V(n, \sigma^2)$.

Some potentially useful relations: if $z \sim N(\mu, \sigma^2)$, then $az \sim N(a\mu, \sigma^2 a^2)$ for a fixed a. If $z_1 \sim N(\mu_1, \sigma_1^2)$ and $z_2 \sim N(\mu_2, \sigma_2^2)$ and they are independent, then $\mathrm{Var}(z_1 + z_2) = \sigma_1^2 + \sigma_2^2$.

3. In sequential active learning we are free to choose the next training input x_{n+1}, here within $[1, 4]$, for which we will then receive the corresponding noisy target y_{n+1}, sampled from the underlying model. Suppose we already have $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ and are trying to figure out which x_{n+1} to select. The goal is to choose x_{n+1} so as to help minimize the variance of the predictions $f(x; \hat{w}_n) = \hat{w}_n x$, where $\hat{w}_n$ is the maximum likelihood estimate of the parameter w based on the first n training examples.

   (a) **(2 points)** What is the variance of $f(x; \hat{w}_n)$ due to the noise in the training outputs as a function of x, n, and σ^2 given fixed (already chosen) inputs $x_1, \ldots, x_n$?

   (b) **(2 points)** Which x_{n+1} would we choose (within $[1, 4]$) if we were to next select x with the maximum variance of $f(x; \hat{w}_n)$?

   (c) **(T/F – 2 points)** Since the variance of $f(x; \hat{w}_n)$ only depends on x, n, and σ^2, we could equally well select the next point at random from $[1, 4]$ and obtain the same reduction in the maximum variance.

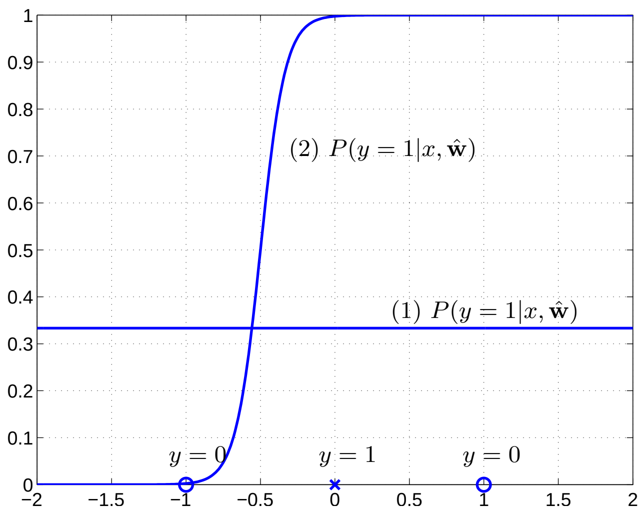


Figure 1: Two possible logistic regression solutions for the three labeled points.

Problem 3

Consider a simple one dimensional logistic regression model

$$P(y=1|x,\mathbf{w}) = g(w_0 + w_1 x)$$

where $g(z) = (1+\exp(-z))^{-1}$ is the logistic function.

1. Figure 3 shows two possible conditional distributions $P(y=1|x,\mathbf{w})$, viewed as a function of x, that we can get by changing the parameters $\mathbf{w}$.

   (a) **(2 points)** Please indicate the number of classification errors for each conditional given the labeled examples in the same figure

   Conditional (1) makes (   ) classification errors

   Conditional (2) makes (   ) classification errors

   (b) **(3 points)** One of the conditionals in Figure 3 corresponds to the maximum likelihood setting of the parameters $\hat{\mathbf{w}}$ based on the labeled data in the figure. Which one is the ML solution (1 or 2)? $\boxed{}$

   (c) **(2 points)** Would adding a regularization penalty $|w_1|^2/2$ to the log-likelihood estimation criterion affect your choice of solution (Y/N)? $\boxed{}$

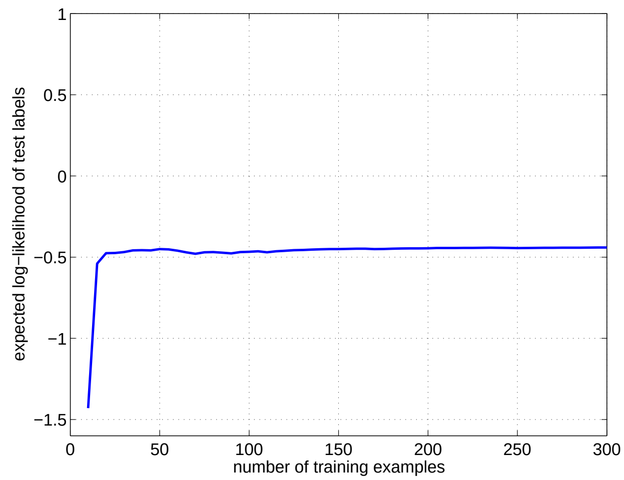


Figure 2: The expected log-likelihood of test labels as a function of the number of training examples.

2. **(4 points)** We can estimate the logistic regression parameters more accurately with more training data. Figure 2 shows the expected log-likelihood of test labels for a simple logistic regression model as a function of the number of training examples and labels. *Mark in the figure* the structural error (SE) and approximation error (AE), where "error" is measured in terms of log-likelihood.

3. **(T/F – 2 points)** In general for small training sets, we are likely to reduce the approximation error by adding a regularization penalty $|w_1|^2/2$ to the log-likelihood criterion.

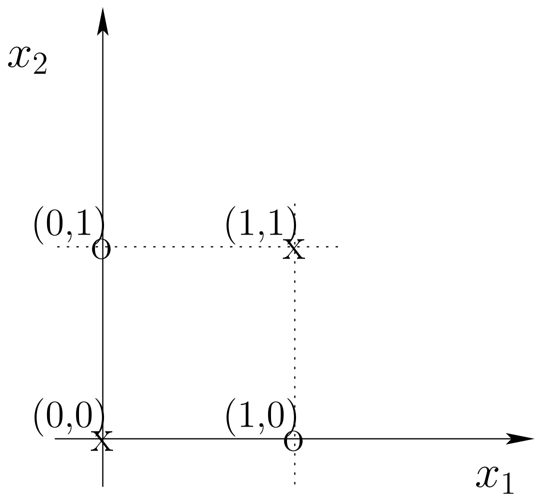


Figure 3: Equally likely input configurations in the training set

Problem 4

Here we will look at methods for selecting input features for a logistic regression model

$$P(y = 1|\mathbf{x}, \mathbf{w}) = g(w_0 + w_1 x_1 + w_2 x_2)$$

The available training examples are very simple, involving only binary valued inputs:

Number of copies	x_1	x_2	y
10	1	1	1
10	0	1	0
10	1	0	0
10	0	0	1

So, for example, there are 10 copies of $\mathbf{x} = [1, 1]^T$ in the training set, all labeled $y = 1$. The correct label is actually a deterministic function of the two features: $y = 1$ if $x_1 = x_2$ and zero otherwise.

We define greedy selection in this context as follows: we start with no features (train only with w_0) and successively try to add new features provided that each addition strictly improves the training log-likelihood. We use no other stopping criterion.

1. **(2 points)** Could greedy selection add either x_1 or x_2 in this case? Answer Y or N.

2. **(2 points)** What is the classification error of the training examples that we could achieve by including both x_1 and x_2 in the logistic regression model?

3. **(3 points)** Suppose we define another possible feature to include, a function of x_1 and x_2. Which of the following features, if any, would permit us to correctly classify all the training examples when used in combination with x_1 and x_2 in the logistic regression model:

( ) $x_1 - x_2$
( ) $x_1 x_2$
( ) x_2^2

4. **(2 points)** Could the greedy selection method choose this feature as the first feature to add when the available features are x_1, x_2 and your choice of the new feature? Answer Y or N.

Problem 5

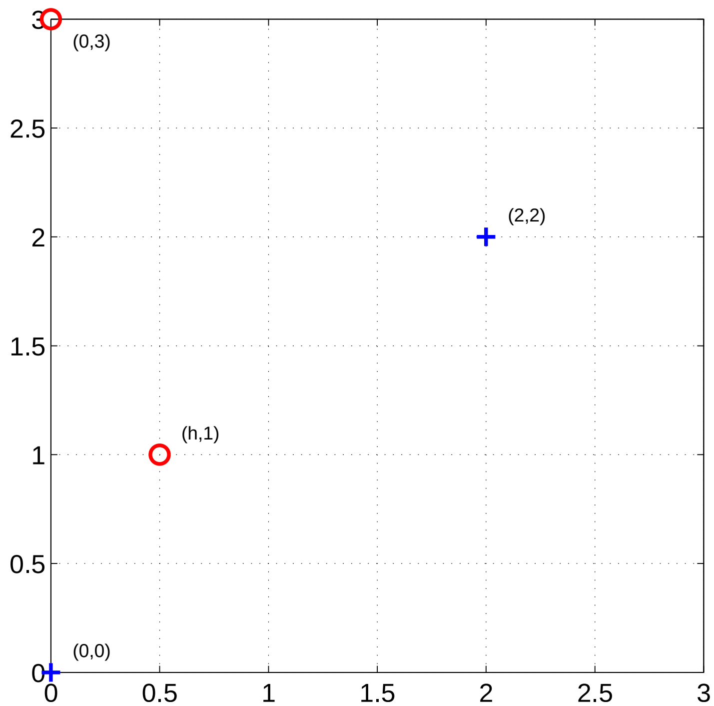


Figure 4: Labeled training examples

Suppose we only have four training examples in two dimensions (see Figure 4):

positive examples at $\mathbf{x}_1 = [0,0]^T, \mathbf{x}_2 = [2,2]^T$ and
negative examples at $\mathbf{x}_3 = [h,1]^T, \mathbf{x}_4 = [0,3]^T$.

where we treat $h \geq 0$ as a parameter.

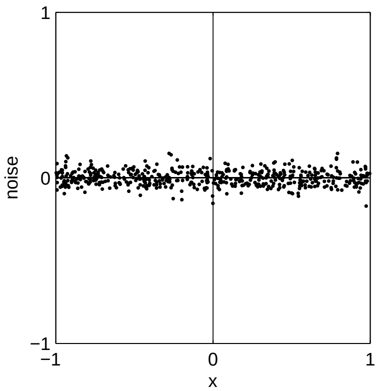
1. **(2 points)** How large can $h \geq 0$ be so that the training points are still linearly separable?

2. **(2 points)** Does the orientation of the maximum margin decision boundary change as a function of h when the points are separable? Answer Y or N.

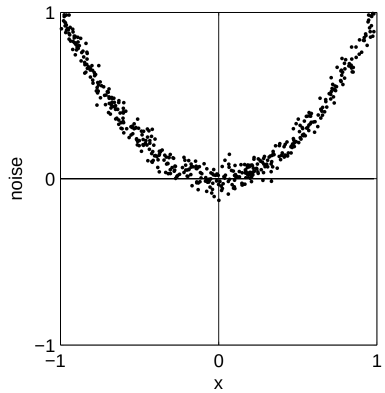
3. **(4 points)** What is the margin achieved by the maximum margin boundary as a function of h?

4. **(3 points)** Assume that $h = 1/2$ (as in the figure) and that we can only observe the x_2-component of the input vectors. Without the other component, the labeled training points reduce to $(0, y = 1)$, $(2, y = 1)$, $(1, y = -1)$, and $(3, y = -1)$. What is the lowest order p of polynomial kernel that would allow us to correctly classify these points?

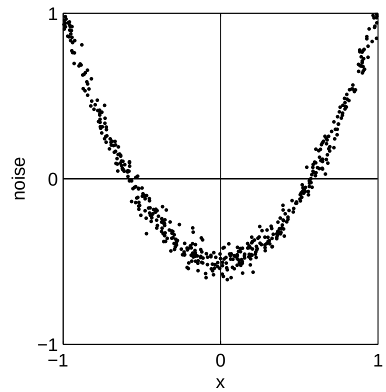
Additional set of figures



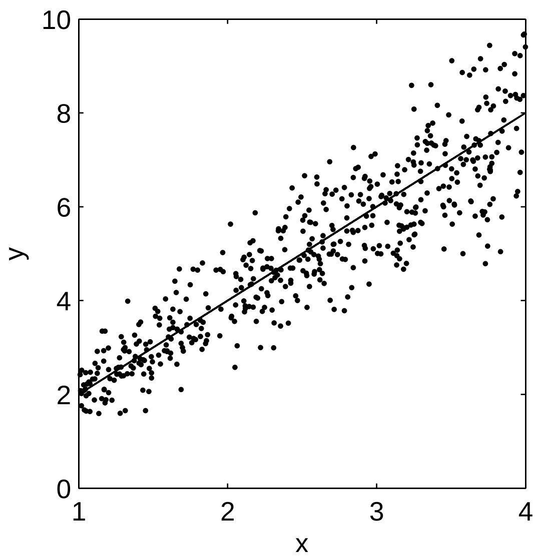
A

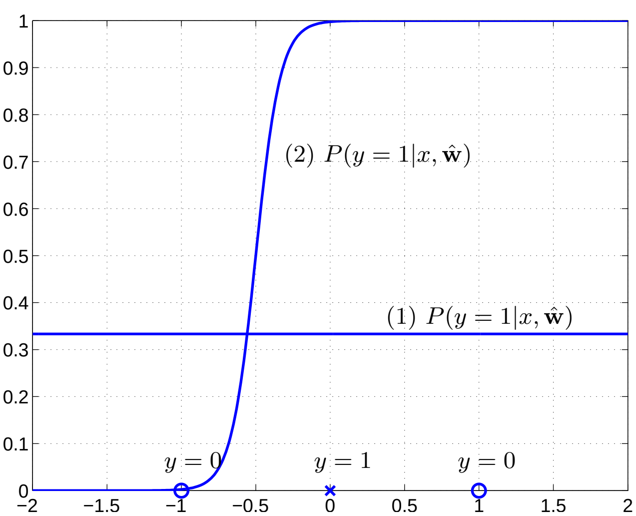


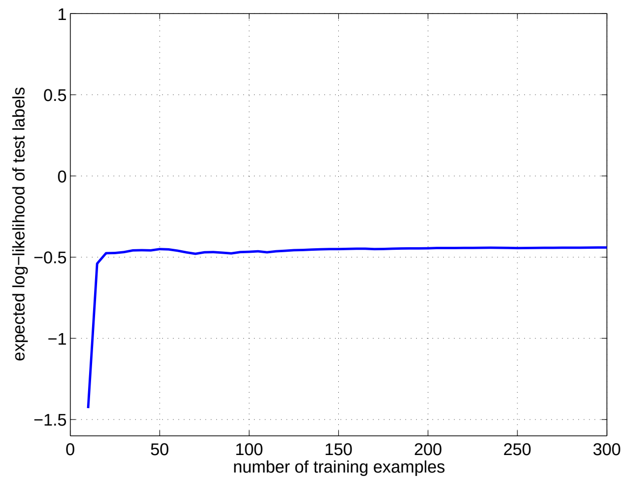
B



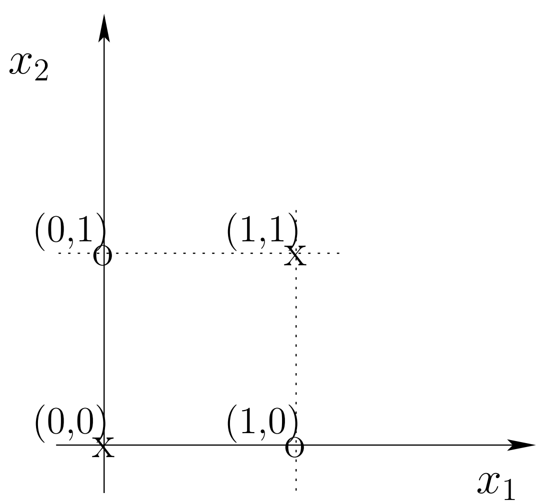
C



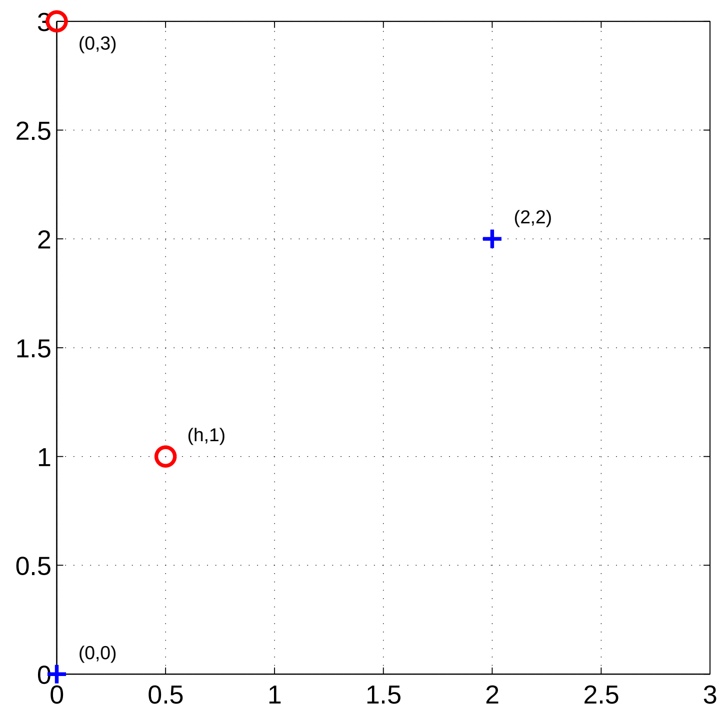




expected log−likelihood of test labels
number of training examples



x_2
x_1
(0,1)
(1,1)
(0,0)
(1,0)



(0,3)
(2,2)
(h,1)
(0,0)