# Machine learning: lecture 12

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# Topics

- Complexity and model selection
  - structural risk minimization
- Complexity, compression, and model selection
  - description length
  - minimum description length principle

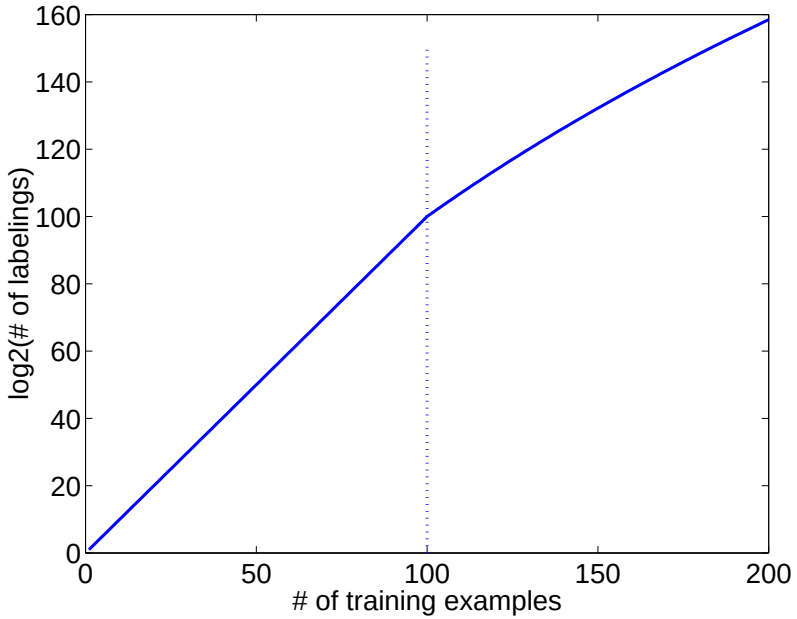
# VC-dimension: review

**Shattering:** A set of classifiers $F$ (e.g., linear classifiers) is said to shatter $n$ points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ if for any possible configuration of labels $y_1, \ldots, y_n$ we can find $h \in F$ that reproduces those labels.

**VC-dimension:** The VC-dimension of a set of classifiers $F$ is the largest number of points that $F$ can shatter (maximized over the choice of the $n$ points).

**Learning:** We don't expect to learn anything until we have more than $d_{VC}$ training examples and labels (this statement will be refined later on).

# The number of labelings




$$n \leq d_{VC}: \quad \#\text{ of labelings } = 2^n$$

$$n > d_{VC}: \quad \#\text{ of labelings } \leq \left(\frac{en}{d_{VC}}\right)^{d_{VC}}$$

# Learning and VC-dimension

- By essentially replacing $\log M$ in the finite case with the log of the number of possible labelings by the set of classifiers over $n$ (really $2n$) points, we get an analogous result:

  **Theorem:** With probability at least $1 - \delta$ over the choice of the training set, for all $h \in F$

$$\mathcal{E}(h) \leq \hat{\mathcal{E}}_n(h) + \epsilon(n, d_{VC}, \delta)$$

  where

$$\epsilon(n, d_{VC}, \delta) = \sqrt{\frac{d_{VC}(\log(2n/d_{VC}) + 1) + \log(1/(4\delta))}{n}}$$

# Model selection

- We try to find the model with the best balance of complexity and fit to the training data
- Ideally, we would select a model from a nested sequence of models of increasing complexity (VC-dimension)

  Model 1, $F_1$ VC-dim $= d_1$
  Model 2, $F_2$ VC-dim $= d_2$
  Model 3, $F_3$ VC-dim $= d_3$

  where $F_1 \subseteq F_2 \subseteq F_3 \subseteq \ldots$
- **Model selection criterion:** find the model (set of classifiers) that achieves the lowest upper *bound* on the expected loss (generalization error):

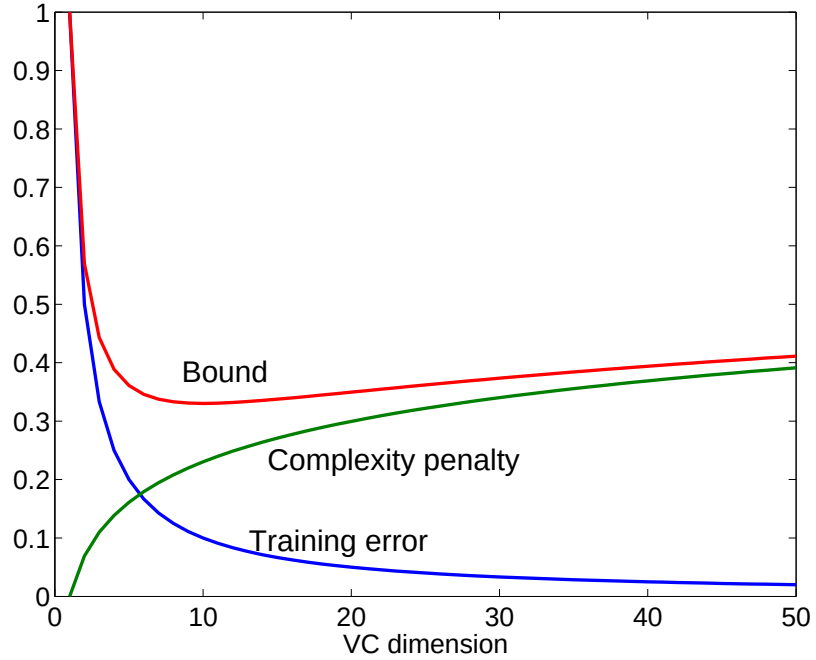
  $$\text{Expected error} \leq \text{Training error} + \text{Complexity penalty}$$

# Structural risk minimization

- We choose the model class $F_i$ that minimizes the upper bound on the expected error:

$$\mathcal{E}(\hat{h}_i) \leq \hat{\mathcal{E}}_n(\hat{h}_i) + \sqrt{\frac{d_i(\log(2n/d_i)+1)+\log(1/(4\delta))}{n}}$$

where $\hat{h}_i$ is the classifier from $F_i$ that minimizes the training error.



# Example

- Models of increasing complexity

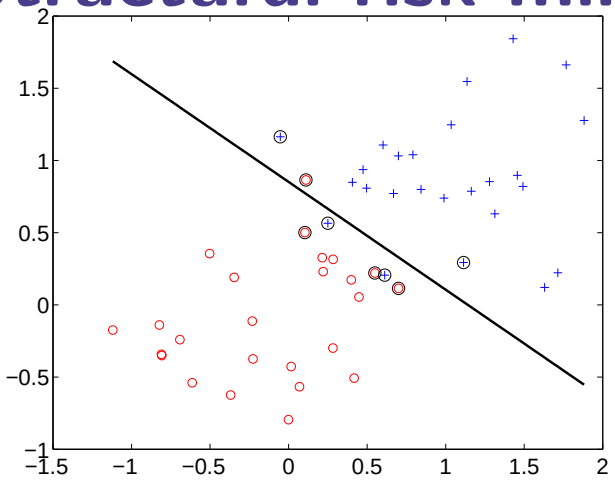
| | |
|---|---|
| Model 1 | $K(\mathbf{x}_1, \mathbf{x}_2) = (1 + (\mathbf{x}_1^T \mathbf{x}_2))$ |
| Model 2 | $K(\mathbf{x}_1, \mathbf{x}_2) = (1 + (\mathbf{x}_1^T \mathbf{x}_2))^2$ |
| Model 3 | $K(\mathbf{x}_1, \mathbf{x}_2) = (1 + (\mathbf{x}_1^T \mathbf{x}_2))^3$ |
| ... | ... |

- These are nested, i.e.,

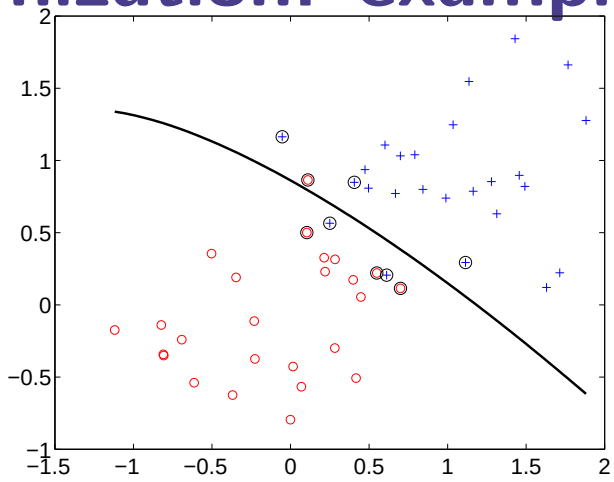
$$F_1 \subseteq F_2 \subseteq F_3 \subseteq \ldots$$

where $F_k$ refers to the set of possible decision boundaries that the model $k$ can represent.

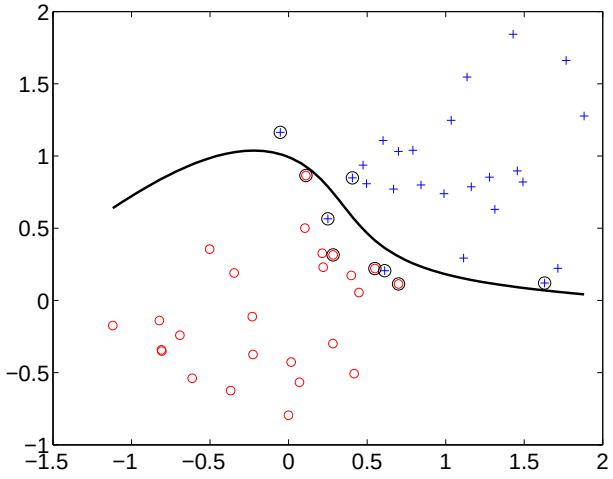
# Structural risk minimization: example



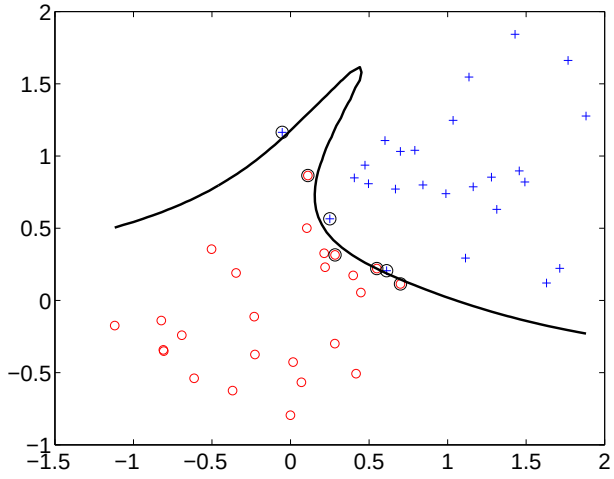
linear



$2^{nd}$ order polynomial



$4^{th}$ order polynomial



$8^{th}$ order polynomial

# Structural risk minimization: example cont'd

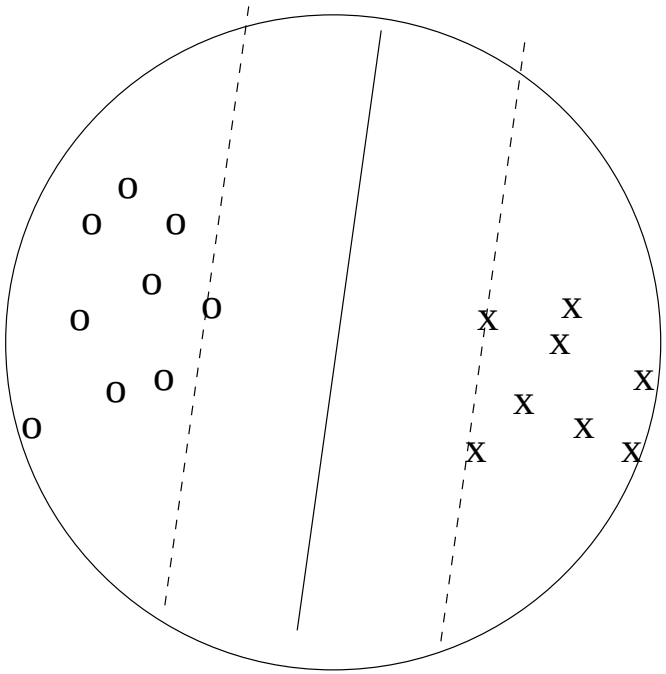
- Number of training examples $n = 50$, confidence parameter $\delta = 0.05$.

| Model | $d_{VC}$ | Empirical fit | $\epsilon(n, d_{VC}, \delta)$ |
|---|---|---|---|
| $1^{st}$ order | 3 | 0.06 | 0.5501 |
| $2^{nd}$ order | 6 | 0.06 | 0.6999 |
| $4^{th}$ order | 15 | 0.04 | 0.9494 |
| $8^{th}$ order | 45 | 0.02 | 1.2849 |

- Structural risk minimization would select the simplest (linear) model in this case.

# Complexity and margin

- The number of possible labelings of points with large margin can be dramatically less than the (basic) VC-dimension would imply



- The set of separating hyperplaces which attain margin $\gamma$ or better for examples within a sphere of radius $R$ has VC-dimension bounded by $d_{VC}(\gamma) \leq R^2/\gamma^2$

# Topics

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# Data compression and model selection

- We can alternatively view model selection as a problem of finding the best way of communicating the available data
- Compression and learning:

| | |
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| $y_1 \quad y_2 \quad \ldots \quad y_n$ | ? ? ... ? |
| $x_1 \quad x_2 \quad \ldots \quad x_n$ | $x_1 \quad x_2 \quad \ldots \quad x_n$ |

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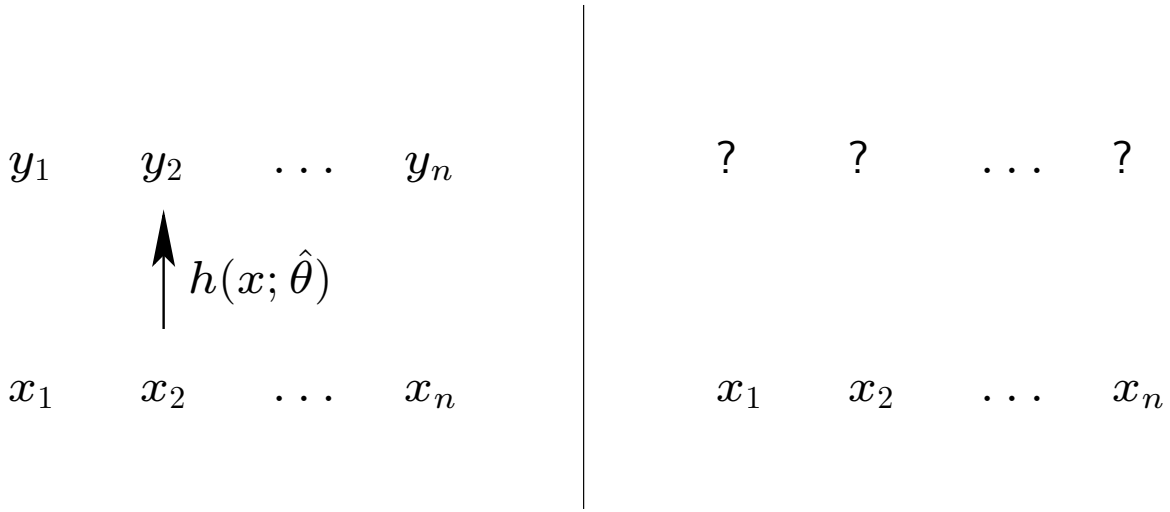

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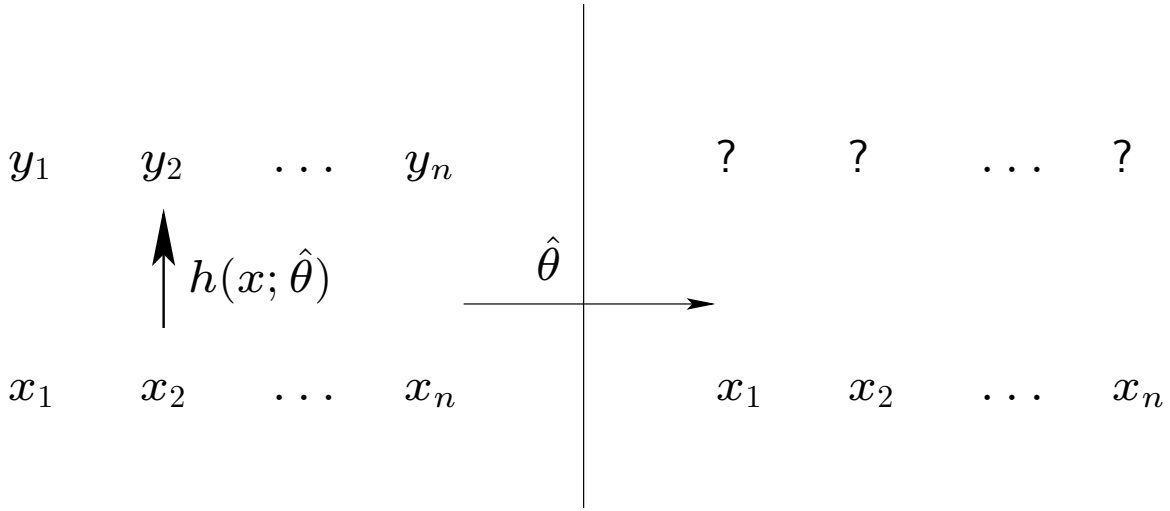

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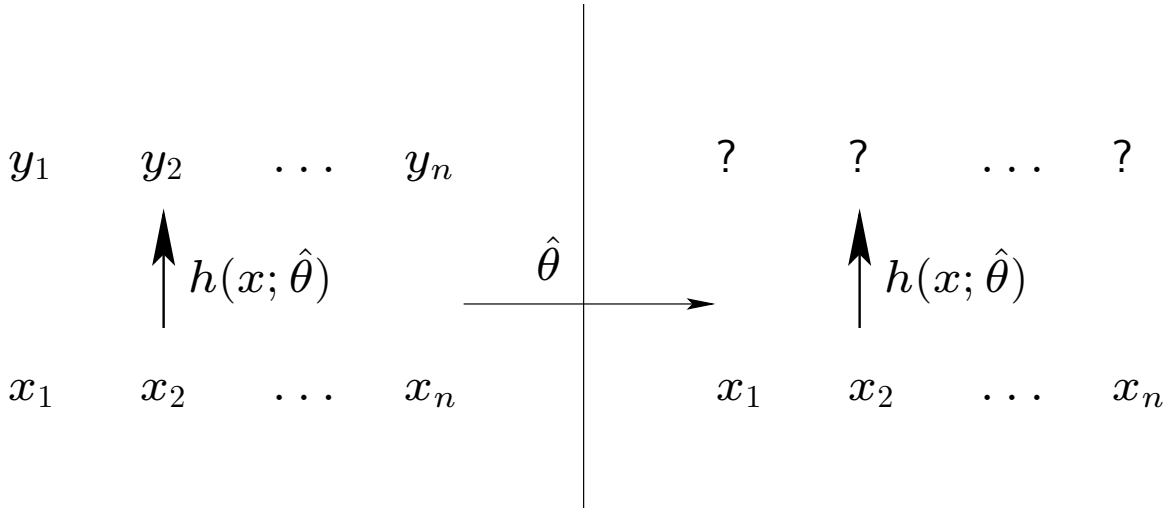

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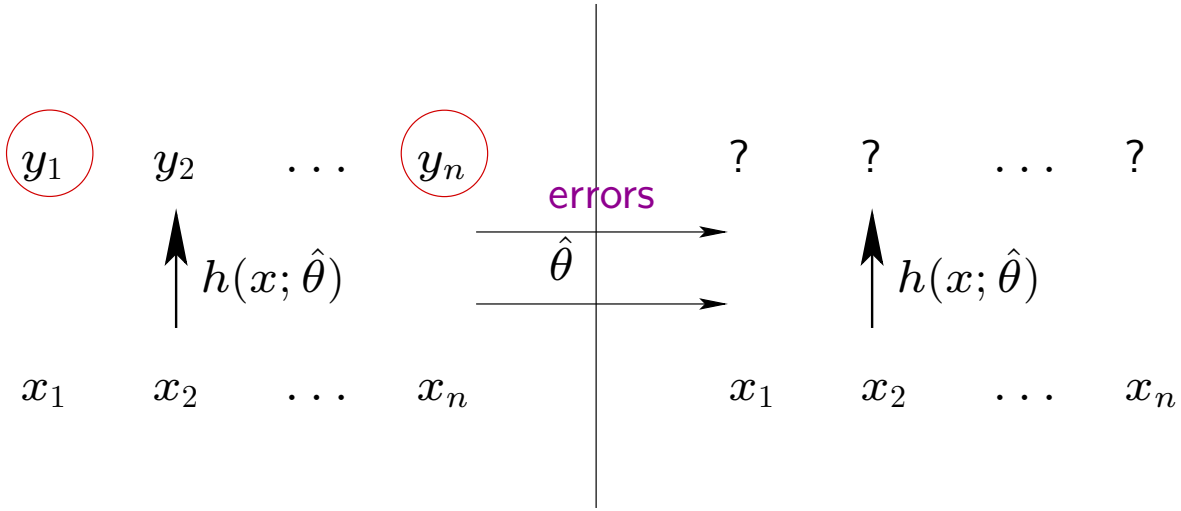

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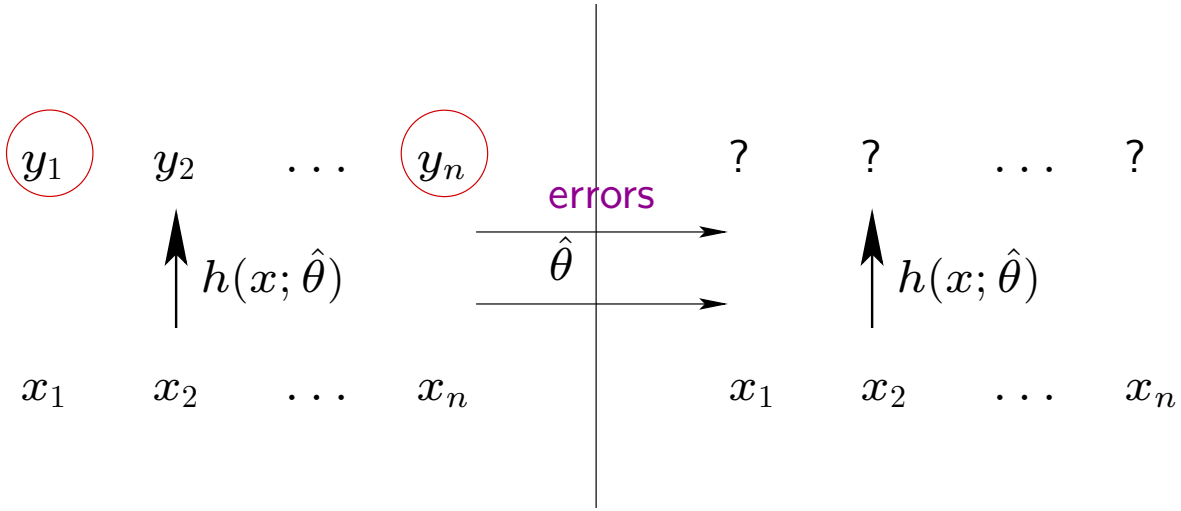


The receiver already knows

– input examples, models we consider

Need to communicate

– model class, parameter estimates, prediction errors

# Compression and sequential estimation

- We don't have to communicate any real valued parameters if we setup the learning problem sequentially

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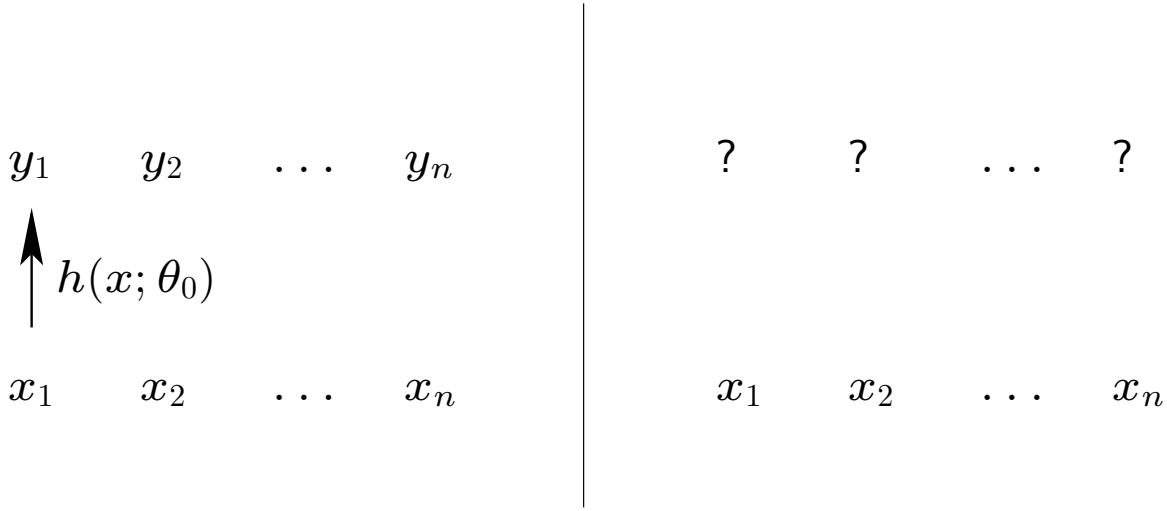


$\theta_0$ : default parameter values

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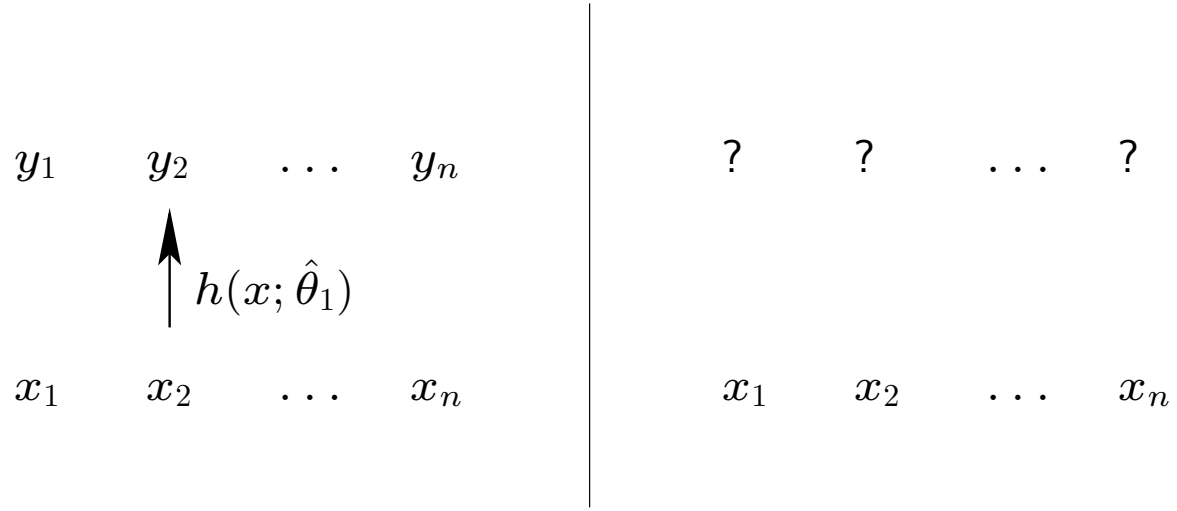


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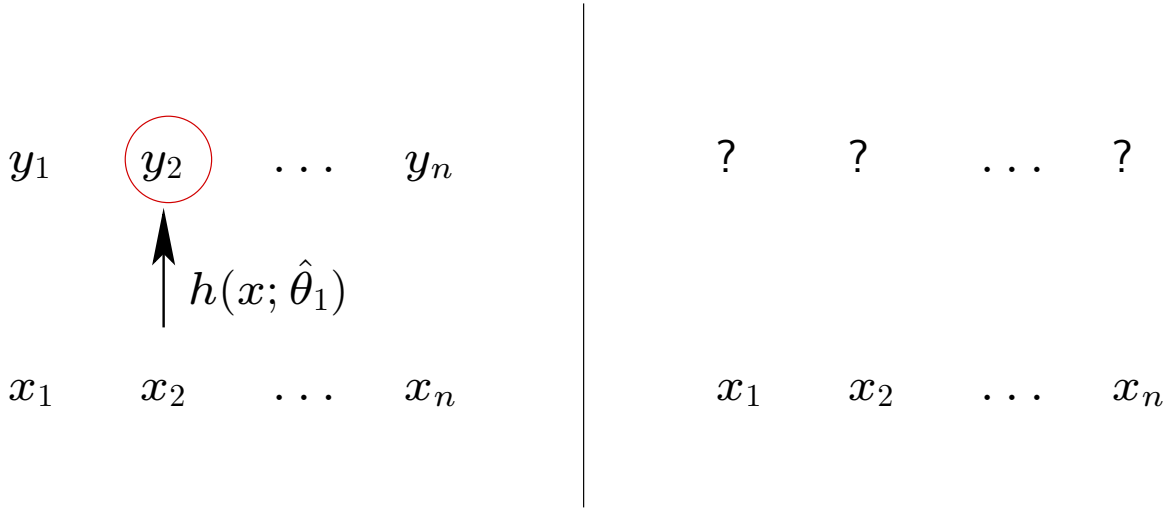


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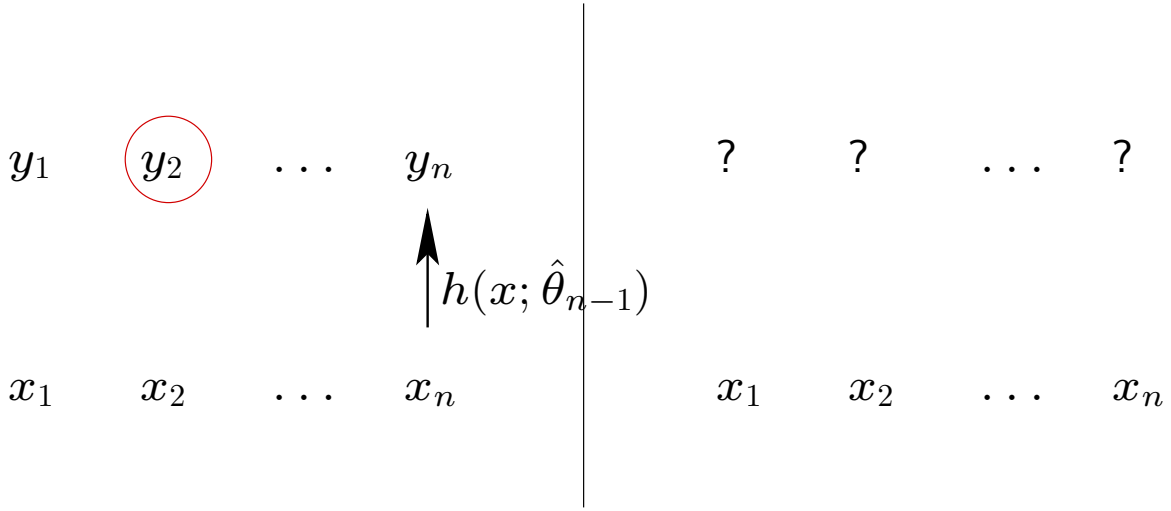


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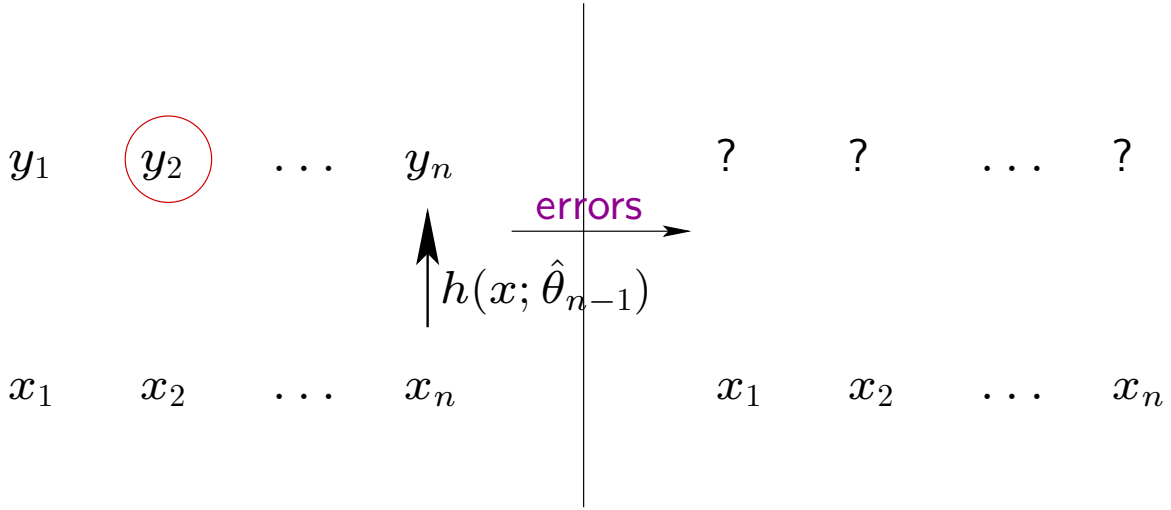


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- we only need to communicate the model class (index) and prediction errors
- but the answer depends on the sequential order

# Probabilistic sequential prediction

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  We predict the first label according to

$$y_1|x_1 \quad : \quad P(y_1|x_1) = \int P(y_1|x_1,\theta)P(\theta)d\theta$$

  and update the prior (posterior)

$$P(\theta|D_1) = \frac{P(\theta)P(y_1|x_1,\theta)}{P(y_1|x_1)}$$

  where $D_1 = \{(x_1, y_1)\}$.

# Probabilistic sequential prediction

- To communicate the labels effectively we need to cast the problem in probabilistic terms
- Suppose we define a model $P(y|x,\theta), \theta \in \Theta$ and prior $P(\theta)$, both known to the receiver

  We predict the second label according to

$$y_2|x_2 \quad : \quad P(y_2|x_2, D_1) = \int P(y_2|x_2,\theta) P(\theta|D_1) d\theta$$

  and again update the posterior

$$P(\theta|D_2) = \frac{P(\theta|D_1)P(y_2|x_2,\theta)}{P(y_2|x_2,D_1)}$$

  where $D_2 = \{(x_1,y_1),(x_2,y_2)\}$.

# Probabilistic sequential prediction

- To communicate the labels effectively we need to cast the problem in probabilistic terms
- Suppose we define a model $P(y|x,\theta), \theta \in \Theta$ and prior $P(\theta)$, both known to the receiver

  Finally, we predict the last $n^{th}$ label according to

$$y_n|x_n \quad : \quad P(y_n|x_n, D_{n-1}) = \int P(y_n|x_n,\theta) P(\theta|D_{n-1}) d\theta$$

  where $D_{n-1} = \{(x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\}$.

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- To communicate the labels effectively we need to cast the problem in probabilistic terms
- Suppose we define a model $P(y|x,\theta), \theta \in \Theta$ and prior $P(\theta)$, both known to the receiver

  Our sequential prediction method defines a probability distribution over all the labels given the examples:

$$P(y_1|x_1)P(y_2|x_2, D_1)\cdots P(y_n|x_n, D_{n-1})$$

  This *does not* depend on the order in which we processed the examples.

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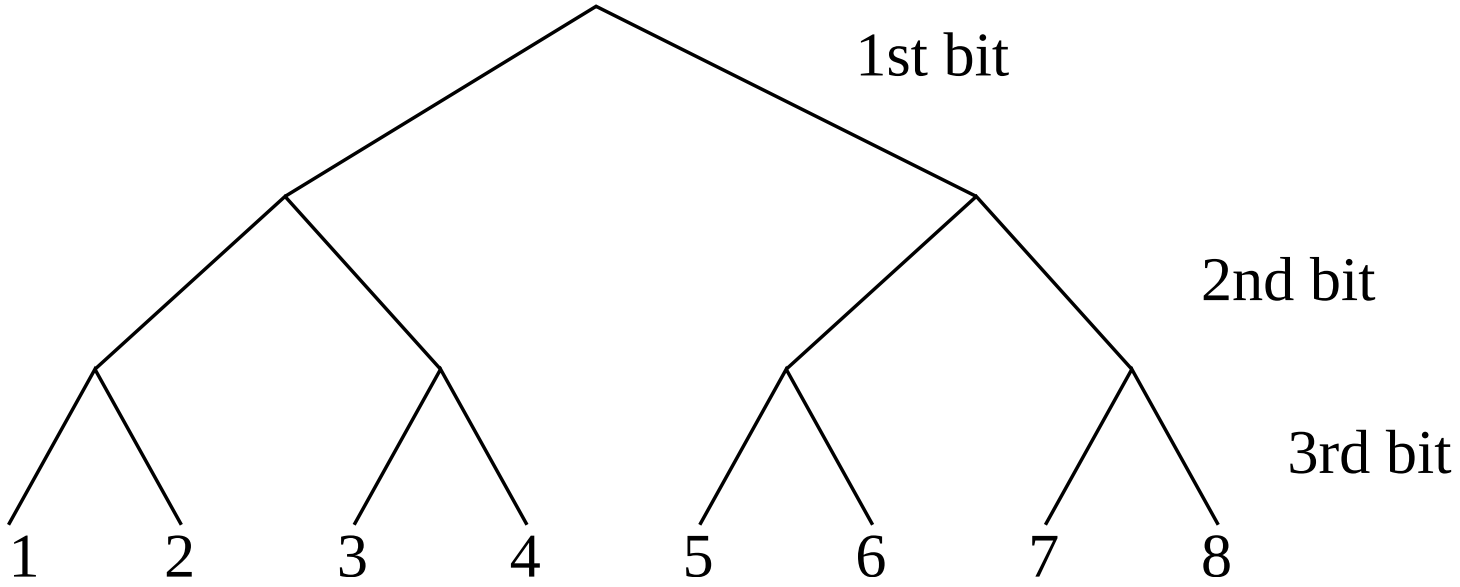
$$= \int P(y_1|x_1,\theta)\cdots P(y_n|x_n,\theta)P(\theta)d\theta$$

  (Bayesian marginal likelihood)

# Description length and probabilities

- It takes $-\log_2 P(y_1, \ldots, y_n)$ bits to communicate $y_1, \ldots, y_n$ according to distribution $P$.

  Example: suppose $y = 1, \ldots, 8$ and each value is equally likely according to $P$




  We need $-\log_2 P(y) = -\log_2(1/8) = 3$ bits to describe each $y$.