Machine learning: lecture 9

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Topics

- Feature selection
  - motivation, examples
  - information value, greedy selection, regularization
- Combination methods
  - forward/backward fitting
  - boosting

Feature selection

- Suppose we consider only a finite collection of possible basis functions, $\phi_i(\mathbf{x})$, $i = 1, \ldots, m$, such as the input components $\phi_i(\mathbf{x}) = x_i$.
- We try to find a small subset S of basis functions that are sufficient to solve the (regression or) classification problem:

$$P(y = 1|\mathbf{x}, \mathbf{w}) = g\big(\, w_0 + \sum_{i=1}^{k} w_i \phi_{s_i}(\mathbf{x})\,\big)$$

where the indexes $S = \{s_1, \ldots, s_k\}$ identify the selected basis functions.

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- There are many ways to find appropriate basis functions:
  - information value
  - greedy selection
  - regularization

Information value

- Let's first try to select the basis functions independently of the classifier, i.e., gauge how "informative" they are in general about the class label.
- Text classification example: $\mathbf{x}$ is a document and the basis functions $\phi_1(\mathbf{x}), \ldots, \phi_m(\mathbf{x})$ are "word indicators"

$$\phi_i(\mathbf{x}) = \begin{cases} 1, & \text{if document } \mathbf{x} \text{ contains word } i \\ 0, & \text{otherwise} \end{cases}$$

  - each document is represented by a binary vector

$$\phi(\mathbf{x}) = \underbrace{[0\;1\;0\;\ldots\;0\;1]^T}_{m \text{ bits}}$$

  - we will derive a score for each feature (bit) based on how much information it contains about the class label

Information value cont'd

- Let's focus on a single feature, e.g., the first one

$$\begin{array}{rccccc} \phi_1: & 0 & 1 & 0 & \ldots & 1 \\ y: & -1 & -1 & 1 & \ldots & 1 \end{array}$$

To assess how the feature values relate to the labels we can calculate the frequency of occurence of different combinations of values: $\hat{P}(y)$, $\hat{P}(\phi_1)$, $\hat{P}(\phi_1, y)$.

For example

$$\hat{P}(\phi_1 = 0, y = 1) = \frac{\#\text{ of docs such that } \phi_1(\mathbf{x}) = 0 \text{ and } y = 1}{n}$$

Information value cont'd

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- The *mutual information* score for each feature is given by:

$$I(\phi_1; y) = \sum_{\phi_1 \in \{0,1\}} \sum_{y \in \{-1,1\}} \hat{P}(\phi_1, y) \log_2 \frac{\hat{P}(\phi_1, y)}{\hat{P}(y)\hat{P}(\phi_1)}$$

  This score is zero if the label is independent of the feature value; large (but ≤ 1) if they are deterministically related.

Selection by information value

- We rank the features according to their mutual information scores (in the descending order of the score):

$$I(\phi_1; y) = \sum_{\phi_1 \in \{0,1\}} \sum_{y \in \{-1,1\}} \hat{P}(\phi_1, y) \log_2 \frac{\hat{P}(\phi_1, y)}{\hat{P}(y)\hat{P}(\phi_1)}$$

  – how many features to include?
  – redundant features?
  – coordination among the features?
  – which classifier can make use of these features?

Greedy selection

1. Find s_1 and $\mathbf{w} = [w_0, w_1]^T$ such that

$$P(y = 1|\mathbf{x}, \mathbf{w}) = g\big(\, w_0 + w_1 \phi_{s_1}(\mathbf{x})\,\big)$$

   leads to the best classifier.

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   leads to the best classifier.

2. Find s_2 and $\mathbf{w} = [w_0, w_1, w_2]^T$ such that

$$P(y = 1|\mathbf{x}, \mathbf{w}) = g\big(\, w_0 + w_1 \phi_{s_1}(\mathbf{x}) + w_2 \phi_{s_2}(\mathbf{x})\,\big)$$

   gives the best performing classifier.

Greedy selection

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   leads to the best classifier.

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   gives the best performing classifier.

3. Etc.

- stopping criterion?
- over-fitting?

Regularization

- We can also consider all of the basis functions at once

$$P(y = 1|\mathbf{x}, \mathbf{w}) = g\big(\, w_0 + w_1 \phi_1(\mathbf{x}) + \ldots + w_m \phi_m(\mathbf{x})\,\big)$$

  and introduce a regularization penalty that tries to set the weights to zero unless the corresponding basis functions are useful.

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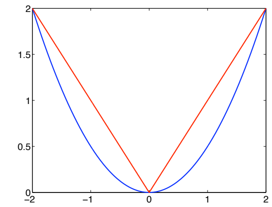
and introduce a regularization penalty that tries to set the weights to zero unless the corresponding basis functions are useful.

$$J(\mathbf{w}; \lambda) = \sum_{i=1}^{n} - \log P(y_i|\mathbf{x}, \mathbf{w}) + \lambda \sum_{i=1}^{m} |w_i|$$

In other words, we regularize the 1-norm (not Euclidean norm) of the weights; w_0 is not penalized

Regularization

- The effect of the regularization penalty depends on its derivative at $w \approx 0$



$w^2/2$ versus $|w|$

$$J(\mathbf{w}; \lambda) = \sum_{i=1}^{n} - \log P(y_i|\mathbf{x}, \mathbf{w}) + \lambda \sum_{i=1}^{m} |w_i|$$

Combination of methods

- Similarly to feature selection we can select simple "weak" classification or regression methods and combine them into a single "strong" method
- Example techniques
  - forward fitting (regression)
  - boosting (classification)

Combination of regression methods

- We want to combine multiple "weak" regression methods into a single "strong" method

$$f(\mathbf{x}) = f(\mathbf{x}; \theta_1) + \ldots + f(\mathbf{x}; \theta_m)$$

- Suppose we are given a family simple regression methods

$$f(\mathbf{x}; \theta) = w\, \phi_k(\mathbf{x})$$

where $\theta = \{k, w\}$ specifies the identity of the basis function as well as the associated weight.

- *Forward-fitting*: sequentially introduce new simple regression methods to reduce the remaining prediction error

Forward fitting cont'd

Simple family: $f(\mathbf{x}; \theta) = w\phi_k(\mathbf{x}), \;\; \theta = \{k, w\}$

- We can fit each new component to reduce the prediction error; in each iteration we solve the same type of estimation problem

Step 1: $\hat{\theta}_1 \leftarrow \arg\min_{\theta} \sum_{i=1}^{n} (y_i - f(\mathbf{x}_i; \theta))^2$

Forward fitting cont'd

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Step 1: $\hat{\theta}_1 \leftarrow \arg\min_{\theta} \sum_{i=1}^{n} (y_i - f(\mathbf{x}_i; \theta))^2$

Step 2: $\hat{\theta}_2 \leftarrow \arg\min_{\theta} \sum_{i=1}^{n} (\underbrace{y_i - f(\mathbf{x}_i; \hat{\theta}_1)}_{\text{error}} - f(\mathbf{x}_i; \theta))^2$

Forward fitting cont'd

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Step 3: $\ldots$

- The resulting combined regression method

$$\hat{f}(\mathbf{x}) = f(\mathbf{x};\hat{\theta}_1) + \ldots + f(\mathbf{x};\hat{\theta}_m)$$

has much lower (training) error.

Forward fitting: example

$f(x;\theta) = wx^k$, where $\theta = \{w, k\}$.