6.891

Computer Vision and Applications

Prof. Trevor Darrell

Lecture 8: Multi-view Geometry

- Instantaneous Essential Matrix
- Fundamental Matrix
- Trifocal Tensor

Readings: F&P Ch 10.

Lecture	Date	Description	Readings	Assignments	Mater
1	2/3	Course Introduction Cameras, Lenses and Sensors	Req: FP 1.1, 2.1, 2.2, 2.3, 3.1, 3.2	PS0 out	
2	2/5	Image Filtering	Req: FP 7.1 - 7.6		
3	2/10	Image Representations: pyramids	Req: FP 7.7, 9.2		
4	2/12	Texture	Req: FP 9.1, 9.3, 9.4	PS0 due	
	2/17	Monday Classes Held (NO LECTURE)			
5	2/19	Color	Req: FP 6.1-6.4	PS1 out	
6	2/24	Local Features			
7	2/26	Multiview Geometry	Req: FP 10	PS1 due	
8	3/2	Multiview Geometry II			
9	3/4	Affine Reconstruction	FP 12	PS2 out	
10	3/9	Projective Reconstruction			
11	3/11	Scene Reconstruction		PS2 due	
12	3/16	Project Previews		EX1 out	
13	3/18	Model-Based Object Recognition		EX1 due	
	3/23-3/25	Spring Break (NO LECTURE)			

Last time

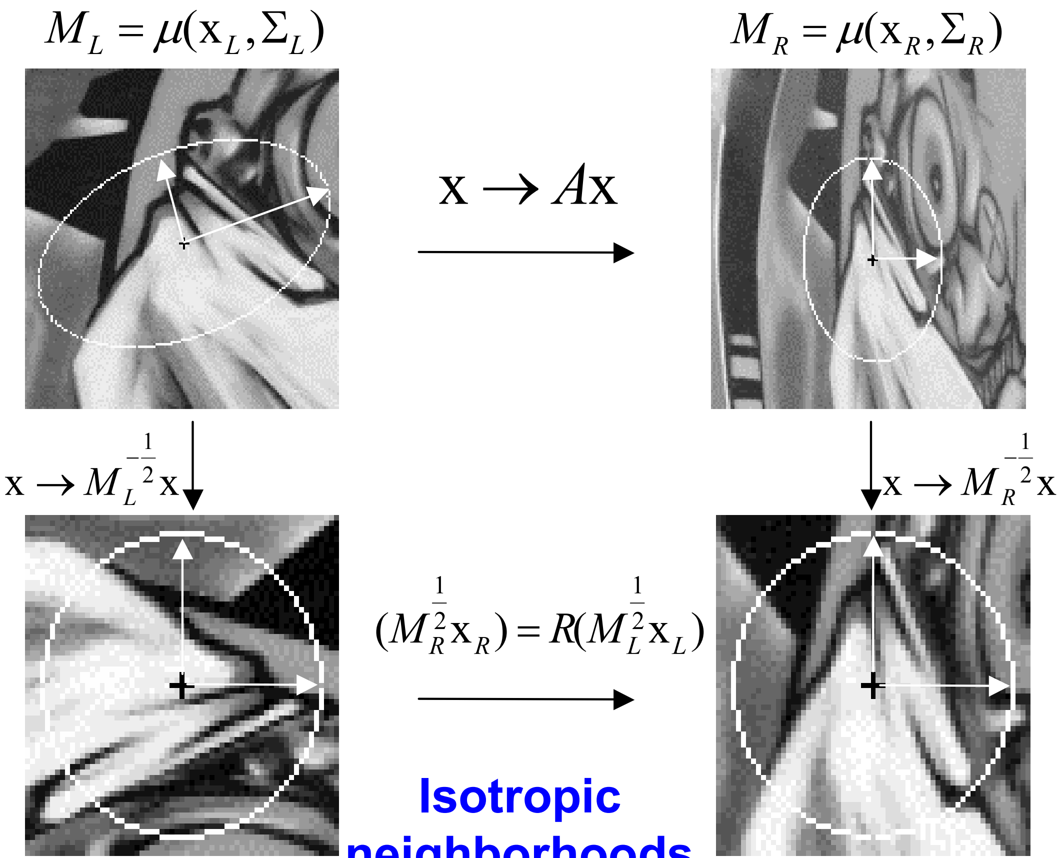
Affine Invariant Interest points [Schmid]

Evaluation of interest points and descriptors [Schmid]

Epipolar geometry and the Essential Matrix

Affine invariant Harris points

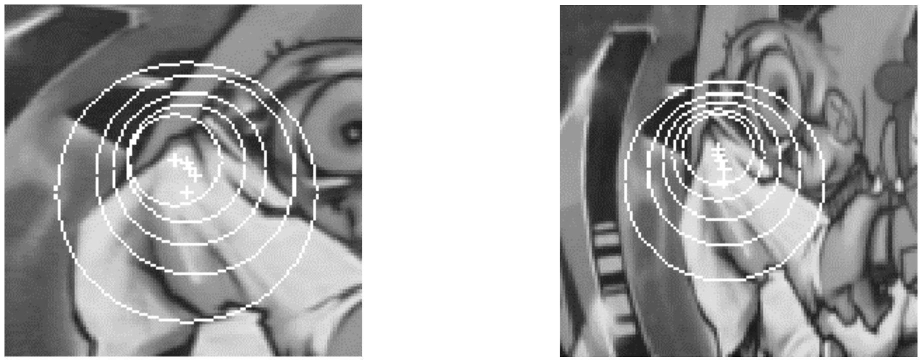
- Theory for affine invariant neighborhood (Lindeberg'94)



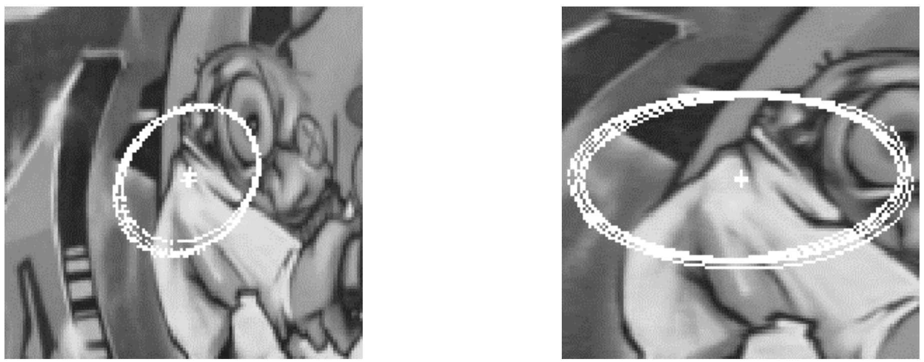
Isotropic neighborhoods related by rotation

Affine invariant Harris points

- Initialization with multi-scale interest points



- Iterative modification of location, scale and neighborhood



Evaluation of affine invariant detectors

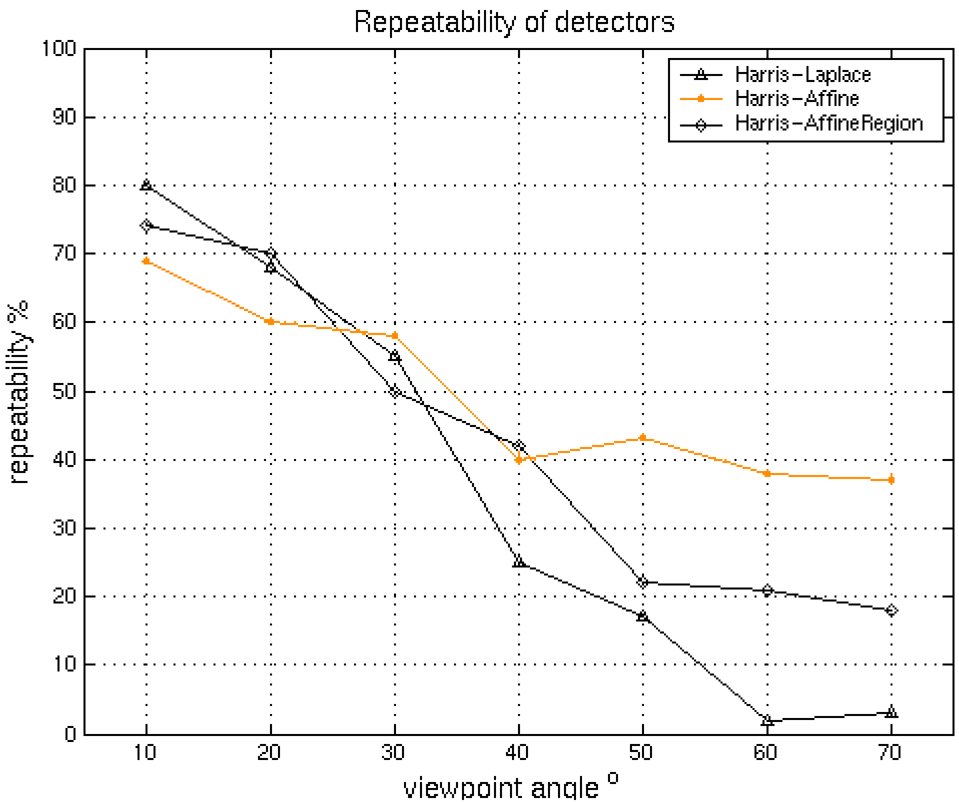
0

40

60

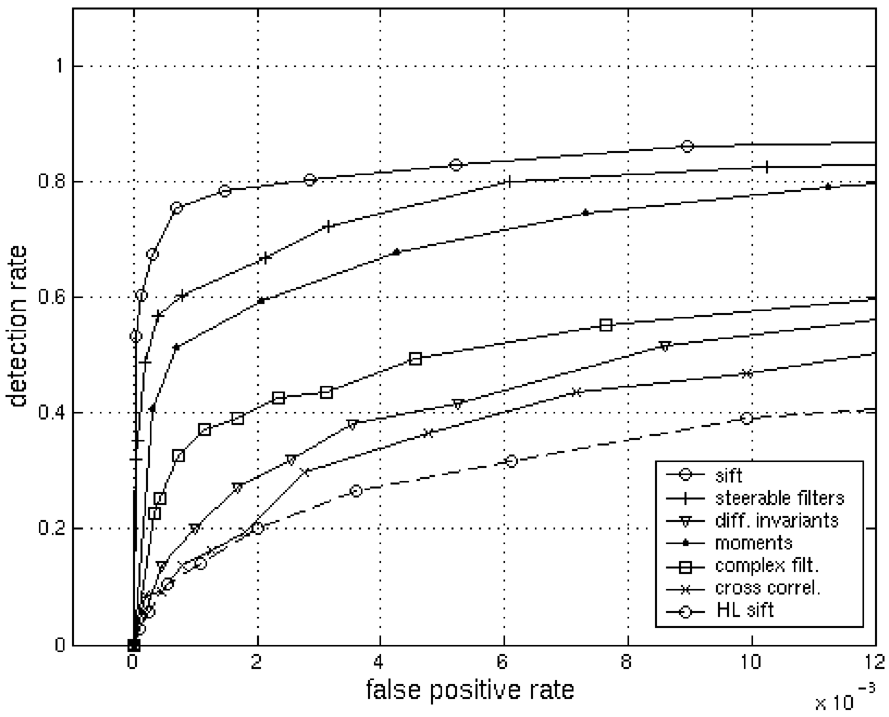
70

repeatability – perspective transformation



Descriptor Evaluation

Viewpoint change (60 degrees)



Harris-Affine (Harris-Laplace)

Epipolar constraint

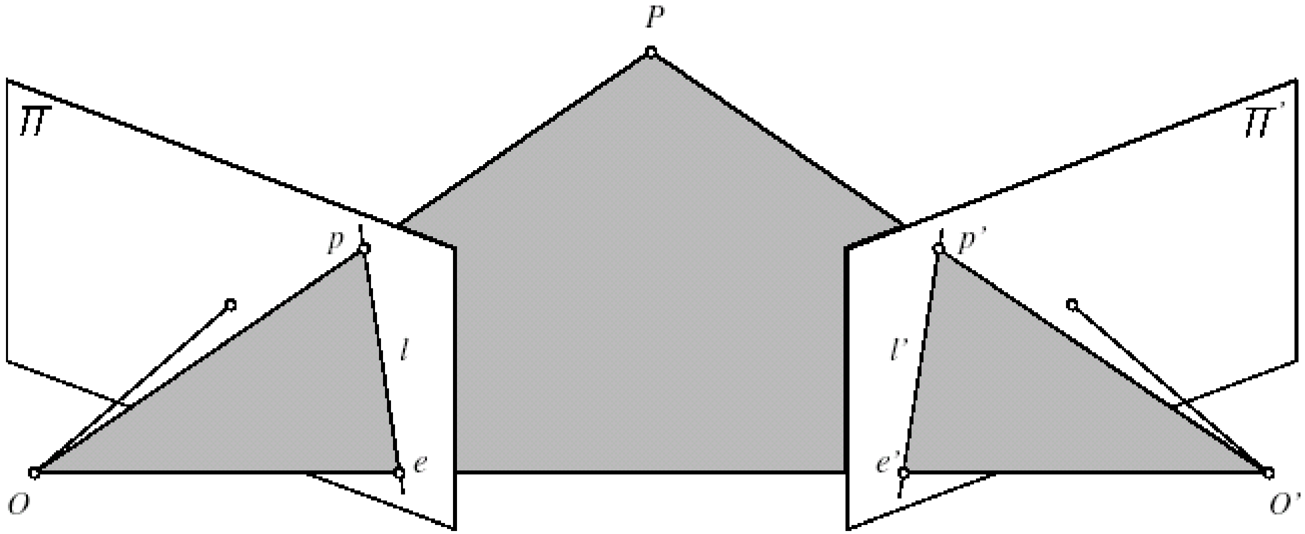


FIGURE 11.1: Epipolar geometry: the point P, the optical centers O and O' of the two cameras, and the two images p and p' of P all lie in the same plane.

All epipolar lines contain epipole, the image of other camera center.

The Essential Matrix

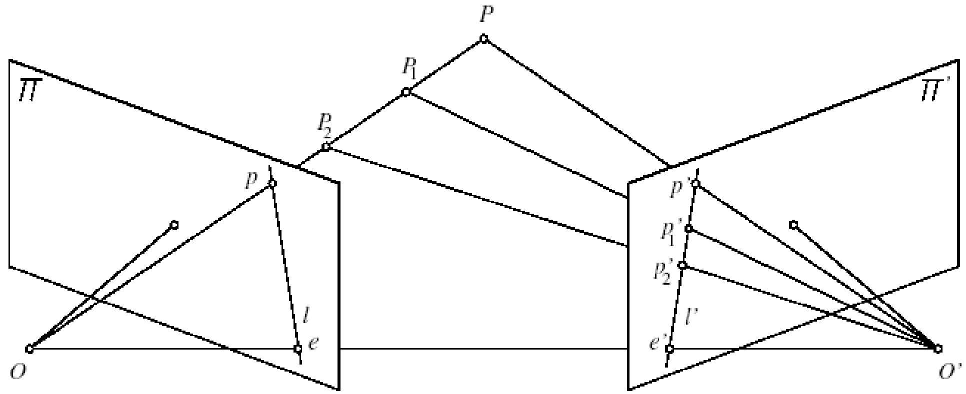
Matrix that relates image of point in one camera to a second camera, given translation and rotation.

Assumes intrinsic parameters are known.

$$\boldsymbol{p} \cdot [\boldsymbol{t} \times (\mathcal{R}\boldsymbol{p}')] = 0$$

$$\varepsilon = [t_x]\Re$$

$$\boldsymbol{p}^T \mathcal{E} \boldsymbol{p}' = 0$$



$$\vec{a} \times \vec{b} = [a_x]\vec{b}$$

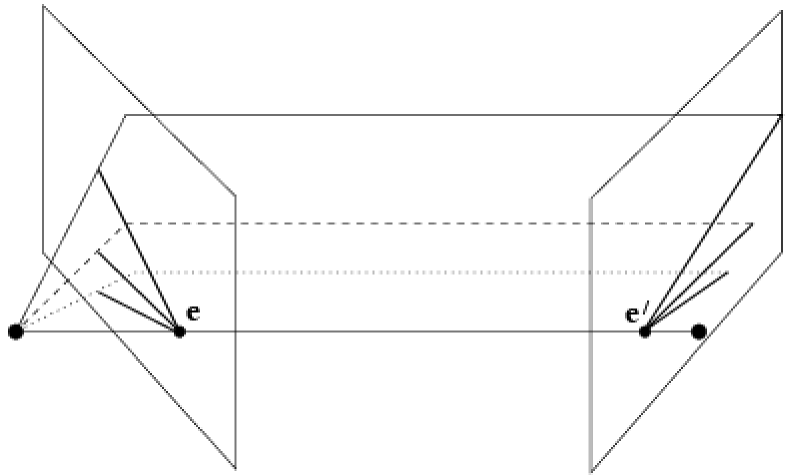
Today

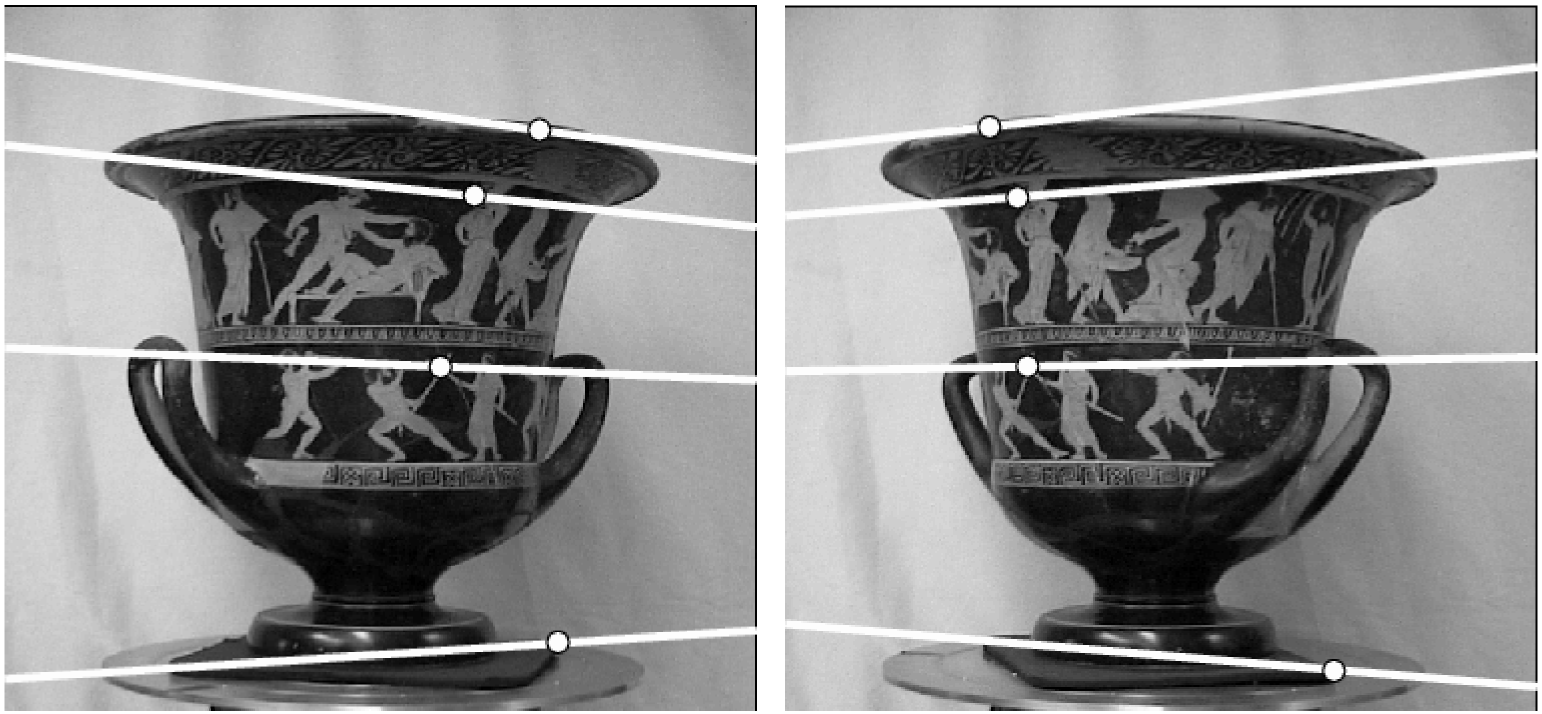
Instantaneous Essential Matricies

Fundamental Matrix and the 8-point algorithm

Tri-focal tensor

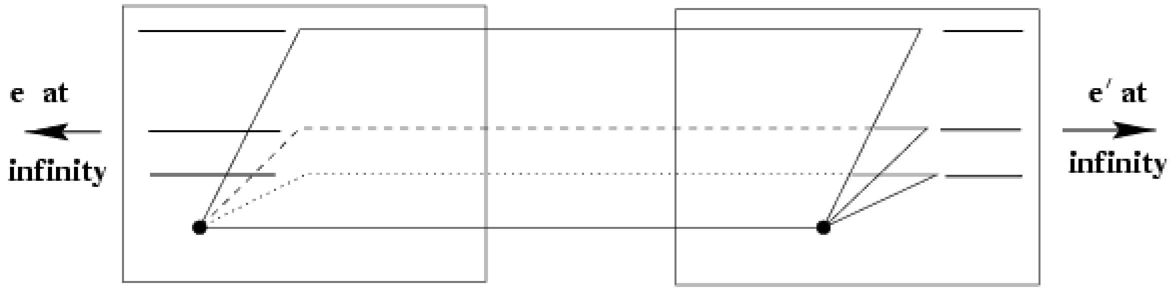
Example: converging cameras

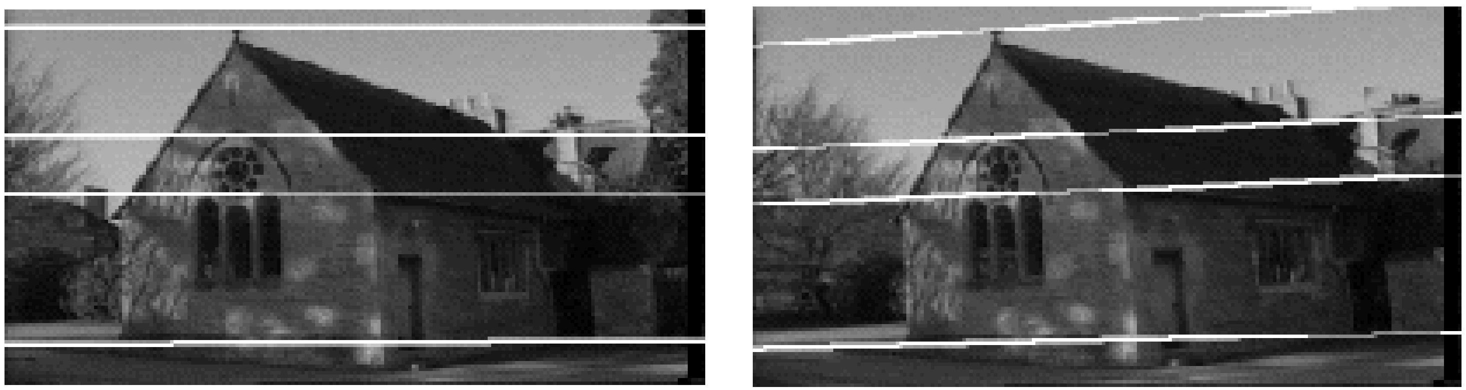




[Image from Marc Pollefeys; www.cs.unc.edu/~marc/mvg/course11.ppt]

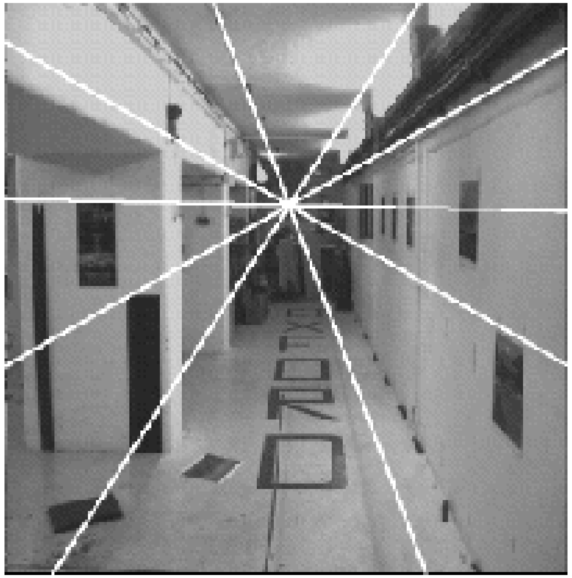
Example: motion parallel with image plane

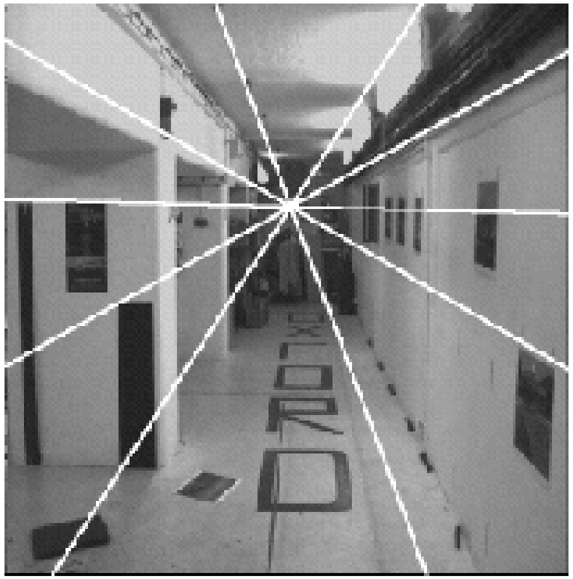


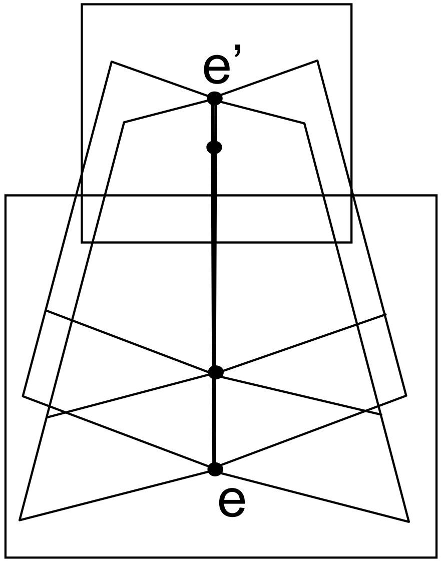


[Image from Marc Pollefeys; www.cs.unc.edu/~marc/mvg/course11.ppt]

Example: forward motion

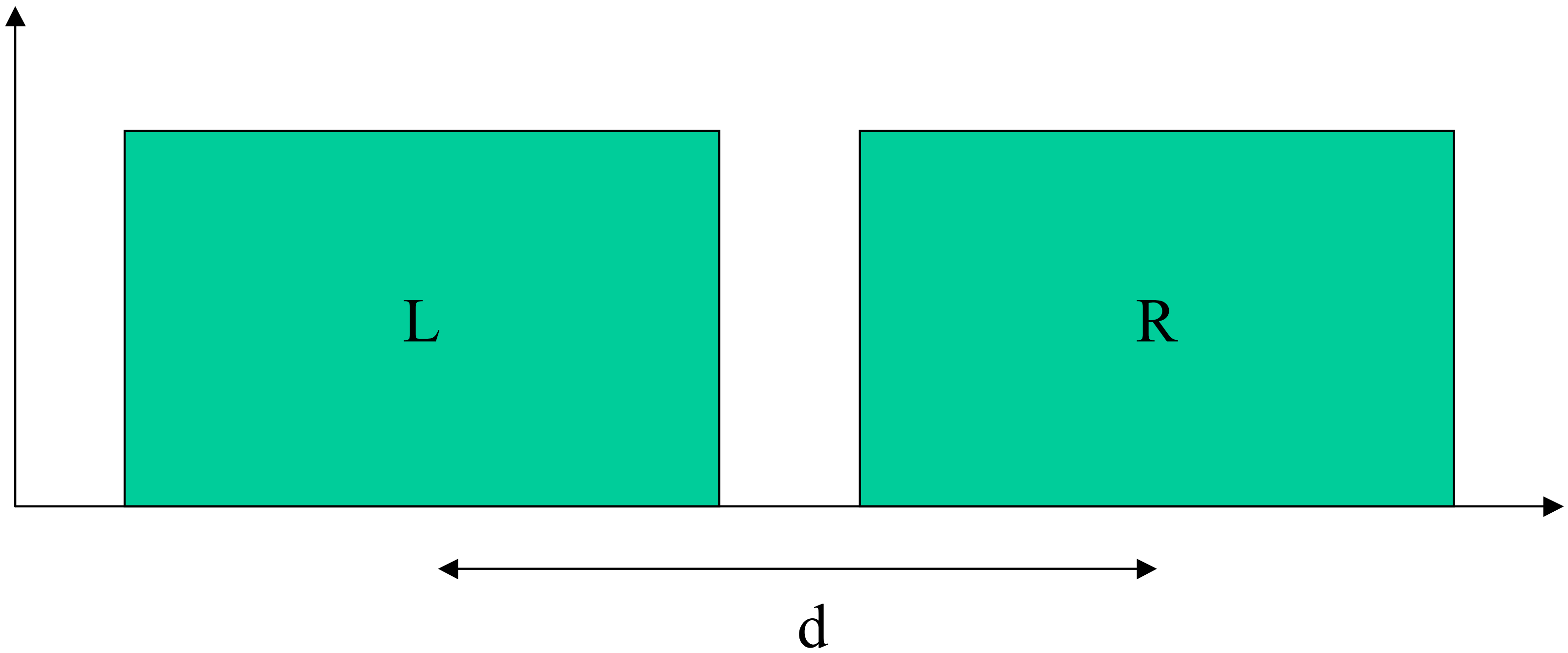






[Image from Marc Pollefeys; www.cs.unc.edu/~marc/mvg/course11.ppt]

Essential matrix for pure tranlation



$$\mathcal{E} = ?$$

Instantaneous Motions

$$t = \delta t \cdot v$$

$$R = I + \delta t[\omega_\times]$$

$$p' = p + \delta t \cdot \dot{p}$$

$$\dot{p} = \begin{bmatrix} V_X \\ V_Y \\ V_Z \end{bmatrix} = \text{ Velocity Vector}$$

$$p^T \mathcal{E} p' = 0$$

$$p^T [v_\times](I + \delta t[\omega_\times])(p + \delta t \cdot \dot{p}) = 0$$

$$v = \begin{bmatrix} V_{T_X} \\ V_{T_Y} \\ V_{T_Z} \end{bmatrix} = \text{ Translational Component of Velocity}$$

$$p^T ([v_\times][\omega_\times])p - (p \times \dot{p}).v = 0$$

$$\omega = \begin{bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{bmatrix} = \text{ Angular Velocity}$$

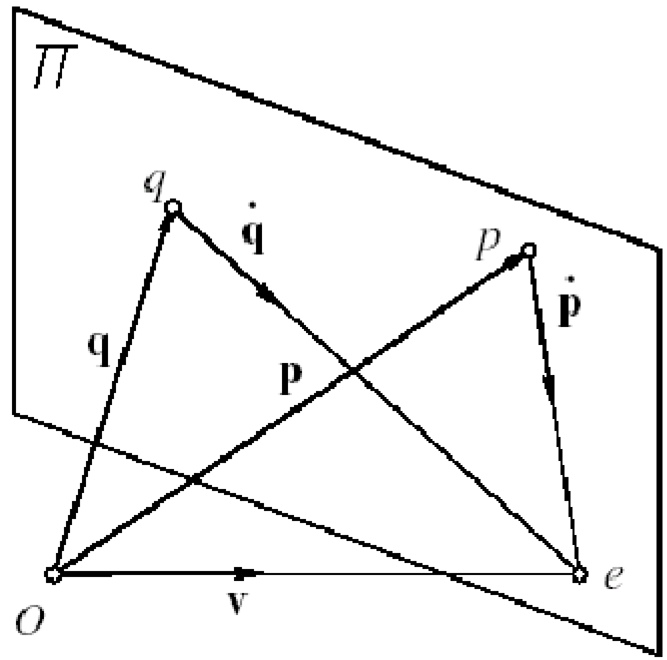
Translating Camera

$$p^T\left([v_\times][\omega_\times]\right)p-\left(p\times\dot{p}\right).v=0$$

$$\omega=0$$

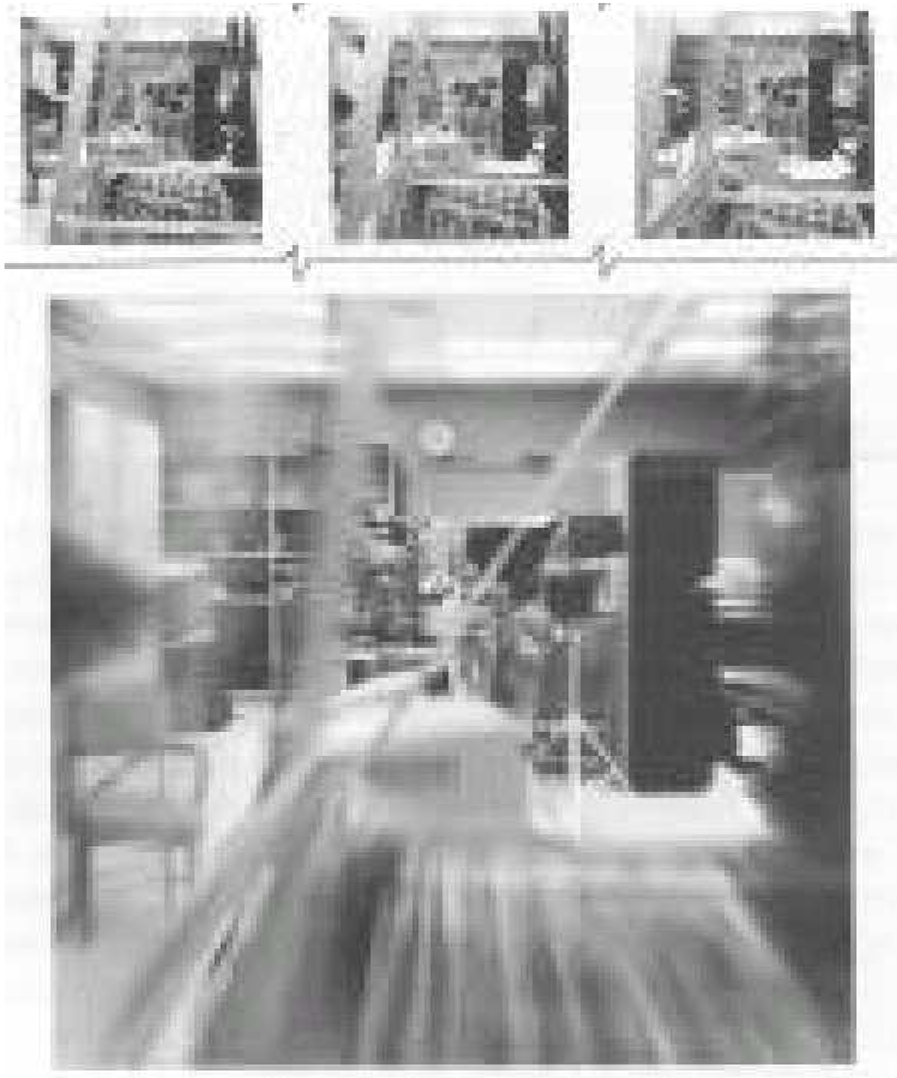
$$\left(p\times\dot{p}\right).v=0$$

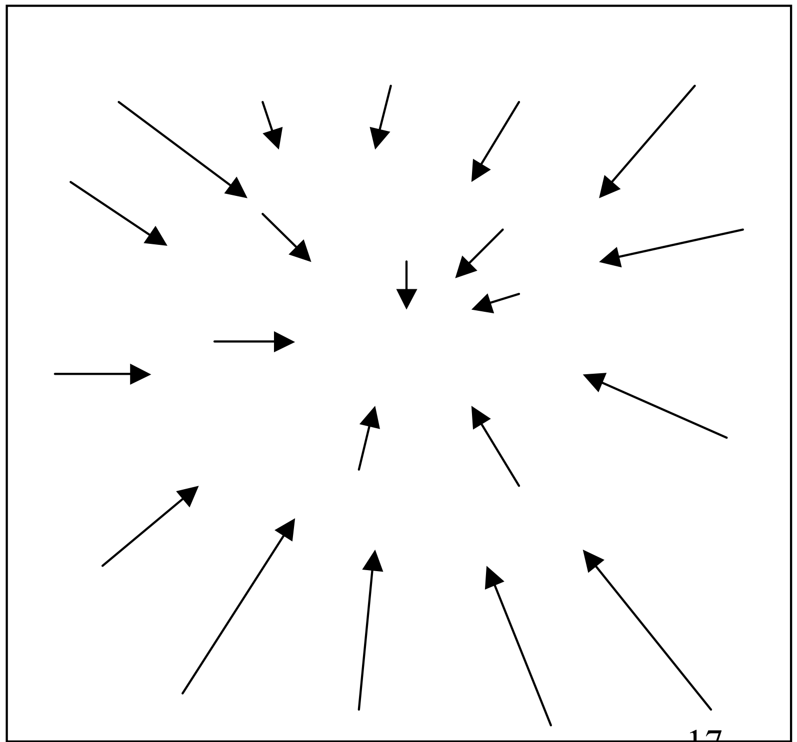
p, $\dot{p}$, and v are coplanar



Focus of expansion (FOE): Under pure translation, the motion field at every point in the image points toward the focus of expansion

FOE for translating camera

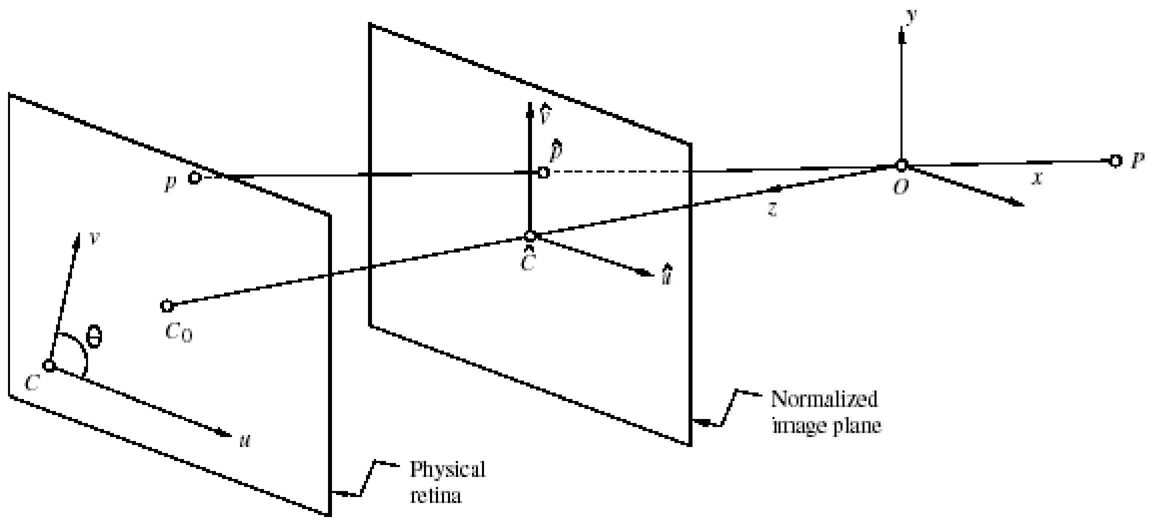




What if calibration is unknown?

Recall calibration eqn:

$$\boldsymbol{p} = \mathcal{K}\hat{\boldsymbol{p}}, \quad \text{where} \quad \boldsymbol{p} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{K} \stackrel{\text{def}}{=} \begin{pmatrix} \alpha & -\alpha\cot\theta & u_0 \\ 0 & \dfrac{\beta}{\sin\theta} & v_0 \\ 0 & 0 & 1 \end{pmatrix}.$$



Fundamental matrix

Essential matrix for points on normalized image plane,

$$\hat{p}^T \mathcal{E} \hat{p}' = 0$$

assume unknown calibration matrix:

$$p = K\hat{p}$$

yields:

$$\boxed{\boldsymbol{p}^T \mathcal{F} \boldsymbol{p}' = 0} \qquad \mathcal{F} = \mathcal{K}^{-T} \mathcal{E} \mathcal{K}'^{-1}$$

Estimating the Fundamental Matrix

$$\boldsymbol{p}^T \mathcal{F} \boldsymbol{p}' = 0$$

Each point correspondence can be expressed as a single linear equation

$$(u, v, 1) \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0$$

Estimating the Fundamental Matrix

$$\boldsymbol{p}^T \mathcal{F} \boldsymbol{p}' = 0$$

Each point correspondence can be expressed as a single linear equation

$$(u, v, 1)\begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}\begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0 \Leftrightarrow (uu', uv', u, vu', vv', v, u', v', 1)\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

Estimating the Fundamental Matrix

How many correspondences are needed to estimate $\mathcal{F}$?

$\mathcal{E}$ has 5 independent parameters up to scale.

In principle $\mathcal{F}$ has 7 independent parameters up to scale, and can be estimated from 7 point correspondences.

Direct, simpler method uses 8 correspondences….

The 8 point algorithm

8 corresponding points, 8 equations.

$$\begin{pmatrix} u_1u_1' & u_1v_1' & u_1 & v_1u_1' & v_1v_1' & v_1 & u_1' & v_1' \\ u_2u_2' & u_2v_2' & u_2 & v_2u_2' & v_2v_2' & v_2 & u_2' & v_2' \\ u_3u_3' & u_3v_3' & u_3 & v_3u_3' & v_3v_3' & v_3 & u_3' & v_3' \\ u_4u_4' & u_4v_4' & u_4 & v_4u_4' & v_4v_4' & v_4 & u_4' & v_4' \\ u_5u_5' & u_5v_5' & u_5 & v_5u_5' & v_5v_5' & v_5 & u_5' & v_5' \\ u_6u_6' & u_6v_6' & u_6 & v_6u_6' & v_6v_6' & v_6 & u_6' & v_6' \\ u_7u_7' & u_7v_7' & u_7 & v_7u_7' & v_7v_7' & v_7 & u_7' & v_7' \\ u_8u_8' & u_8v_8' & u_8 & v_8u_8' & v_8v_8' & v_8 & u_8' & v_8' \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \end{pmatrix} = - \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Invert and solve for $\mathcal{F}$.

(Use more points if available; find least-squares solution to minimize $\sum_{i=1}^{n} (\boldsymbol{p}_i^T \mathcal{F} \boldsymbol{p}_i')^2$)

The 8 point algorithm

$$\boldsymbol{p}^T \mathcal{F} \boldsymbol{p}' = 0$$

is $\mathcal{F}$ (or $\mathcal{E}$) full rank?

No…singular with rank=2.
Has zero eigenvalue corresponding to epipole.

$$\mathcal{F}^T e = 0$$

Improved 8 point algorithm

Enforce rank 2 constraint!

(Also pay attention to numerical conditioning…)

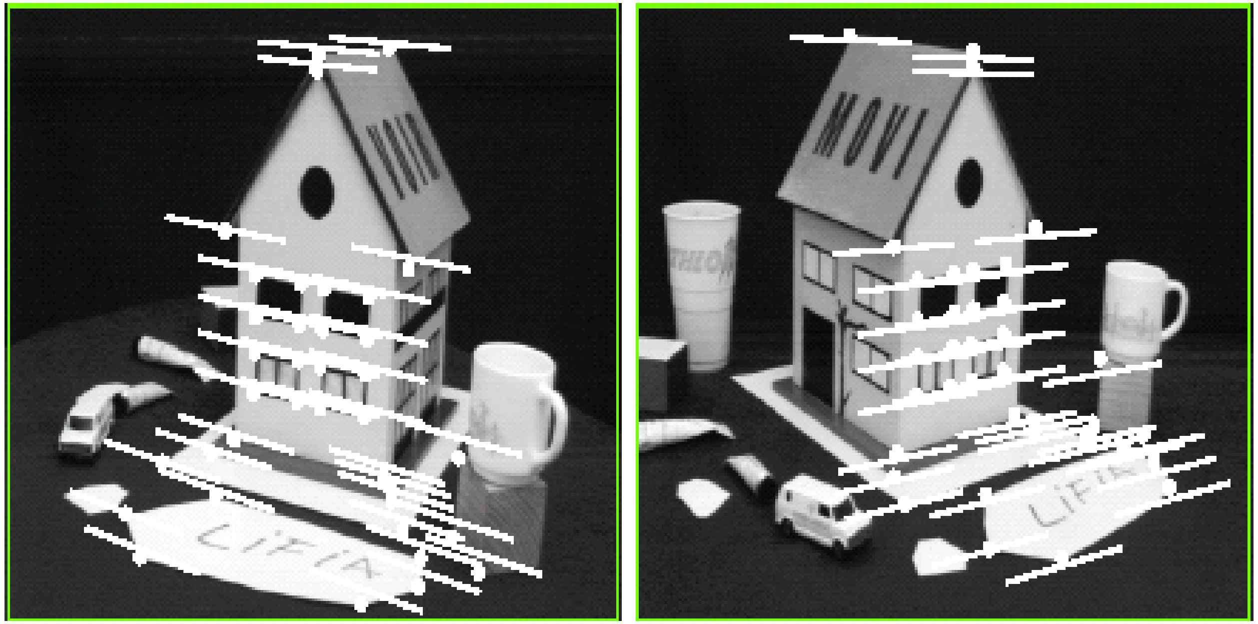
Hartley 1995: use SVD.

1. Transform to centered and scaled coordinates
2. Form least-squares estimate of F
3. Set smallest singular value to zero.

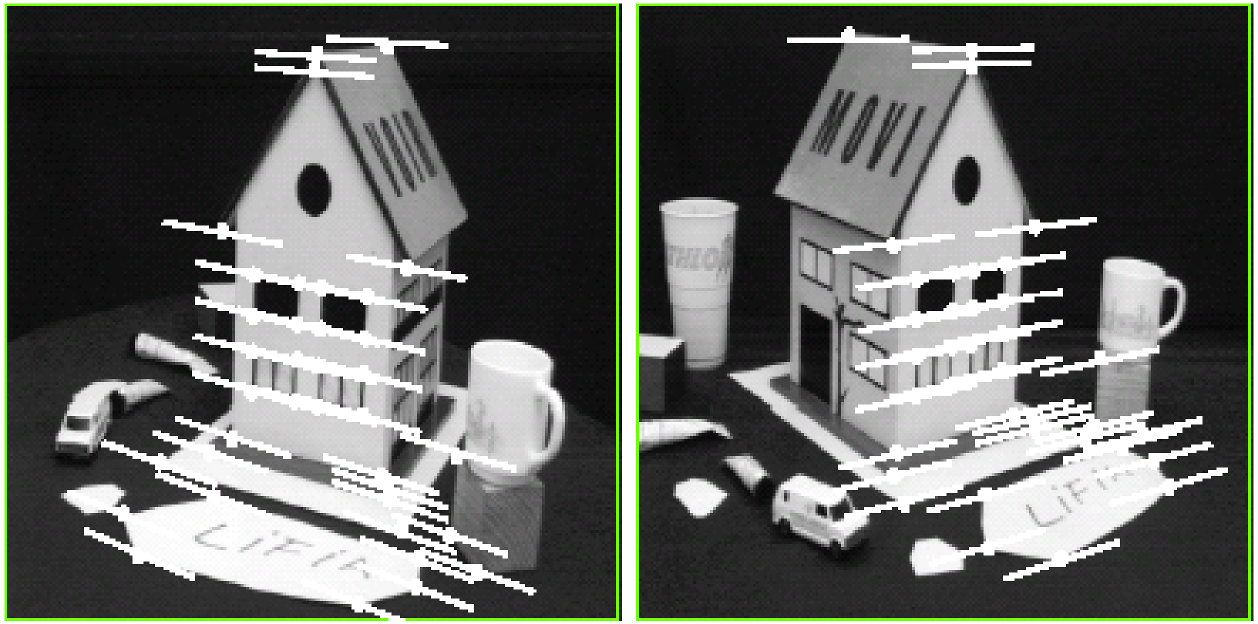
Normalizing the Input Data

- Directly use the pixel coordinates produces bad result
- Normalization method is quite necessary
- Isotropic scaling of the input data:
  - Points are translated to have their centroid at the origin
  - The coordinates are scaled isotropically so that the average distance from the origin to these points is equal to $\sqrt{2}$.

Zhengyou Zhang
Determining the Epipolar Geometry and its Uncertainty: A Review
www.cs.unr.edu/~mircea/Courses/ cs790E/Lectures/zhang2.ppt

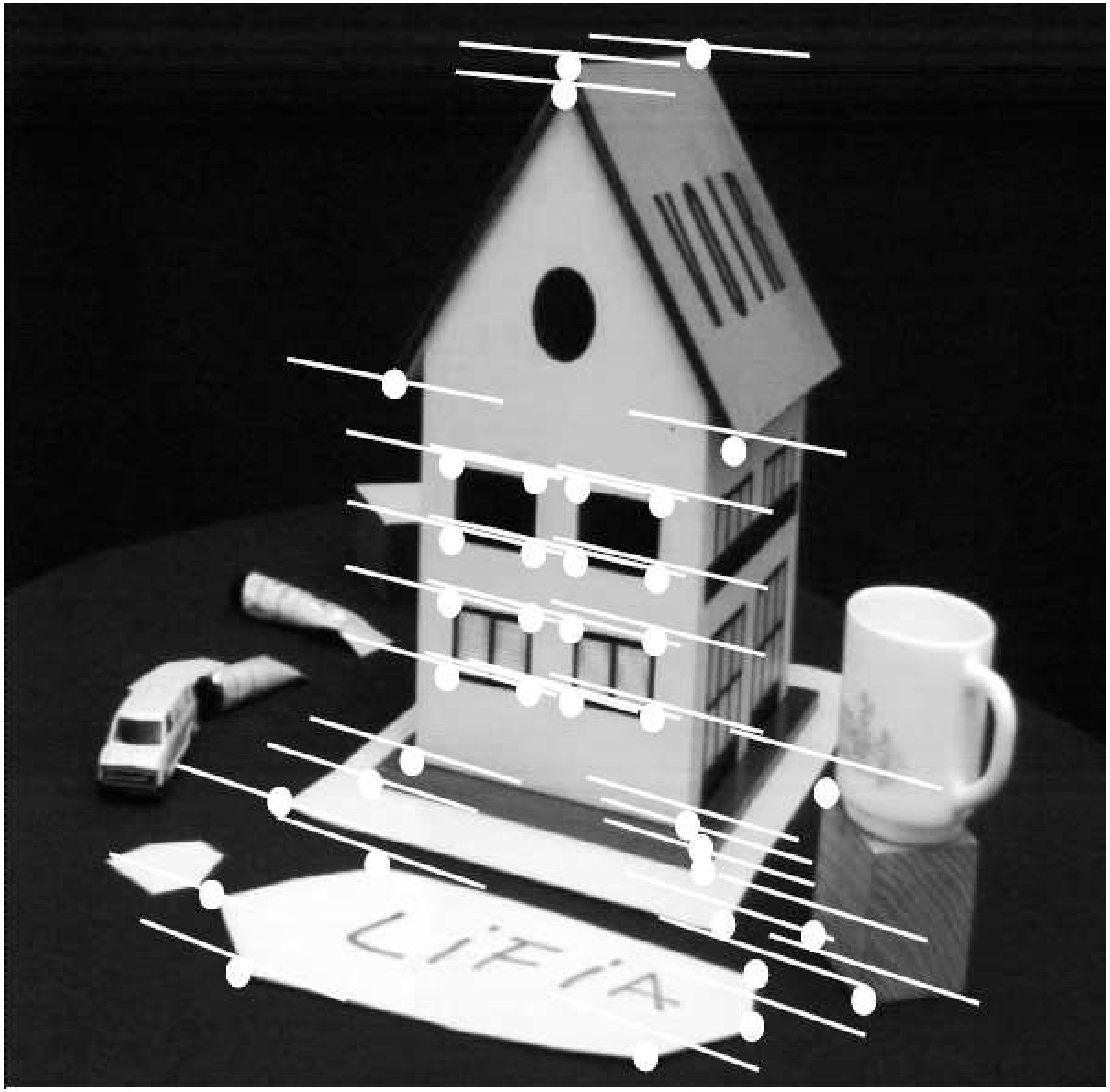


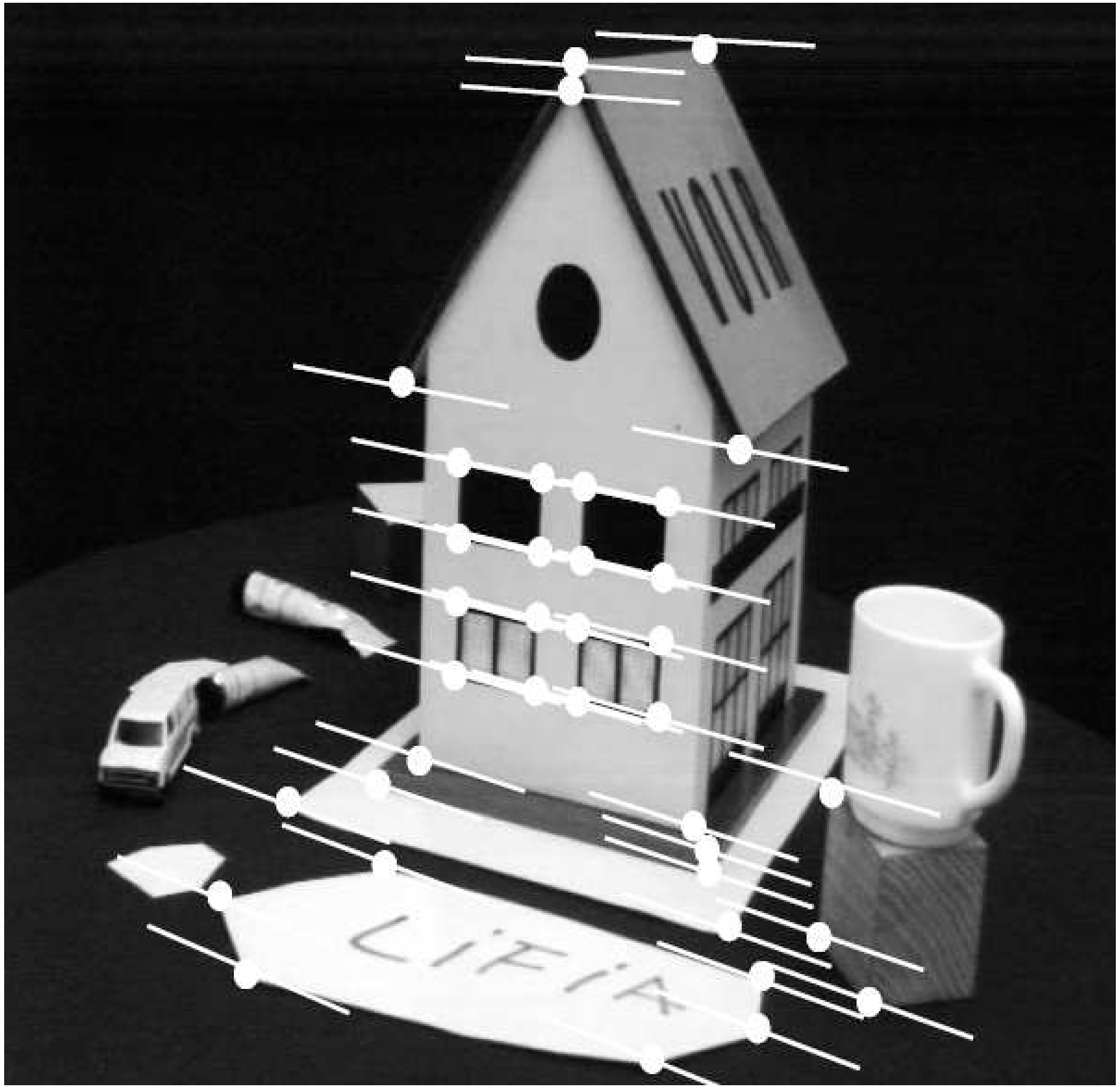
(a)



(b)

	Linear Least Squares	[Hartley, 1995]	[Luong *et al.*, 1993]
Av. Dist. 1	2.33 pixels	0.92 pixel	0.86 pixel
Av. Dist. 2	2.18 pixels	0.85 pixel	0.80 pixel


VOIR
LIFIA


LIFIA

Example of Fundamental Matrix Estimation with Comparison

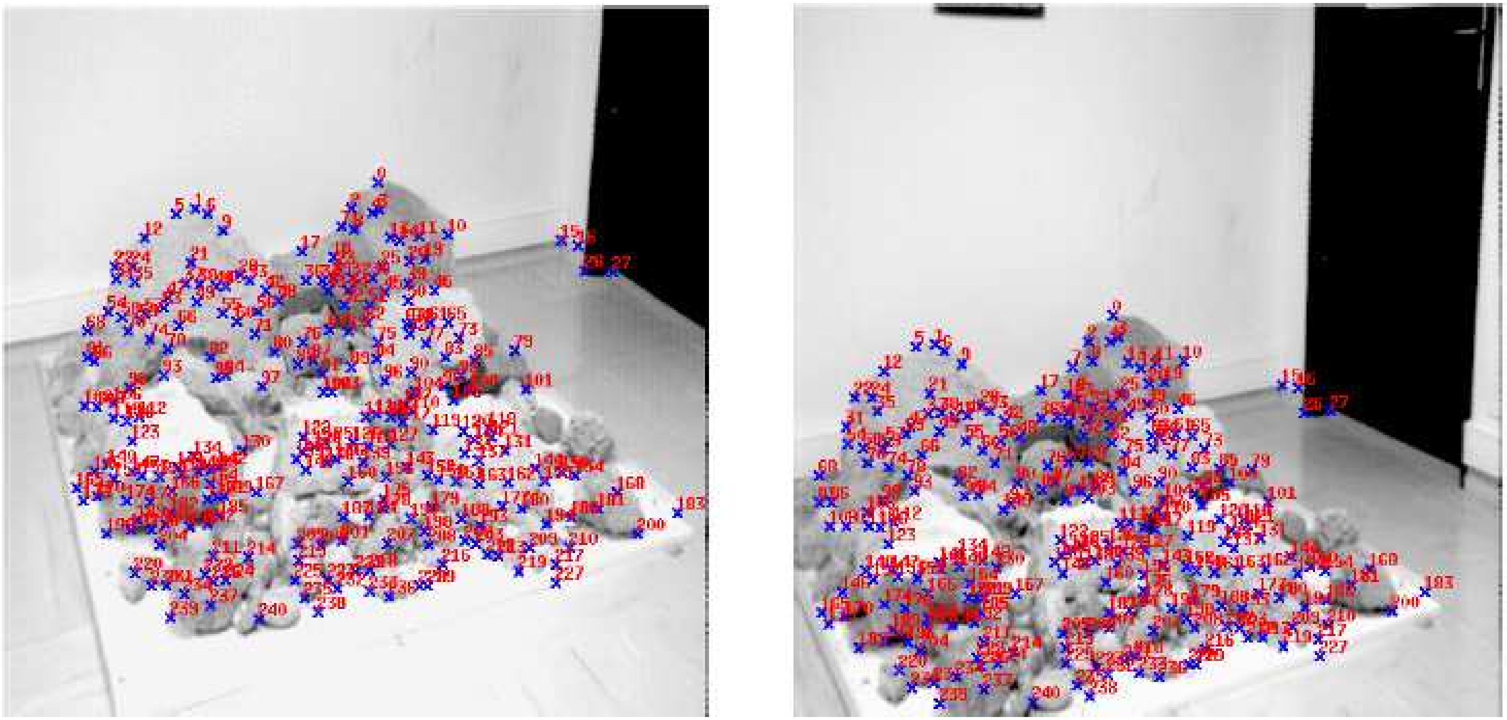


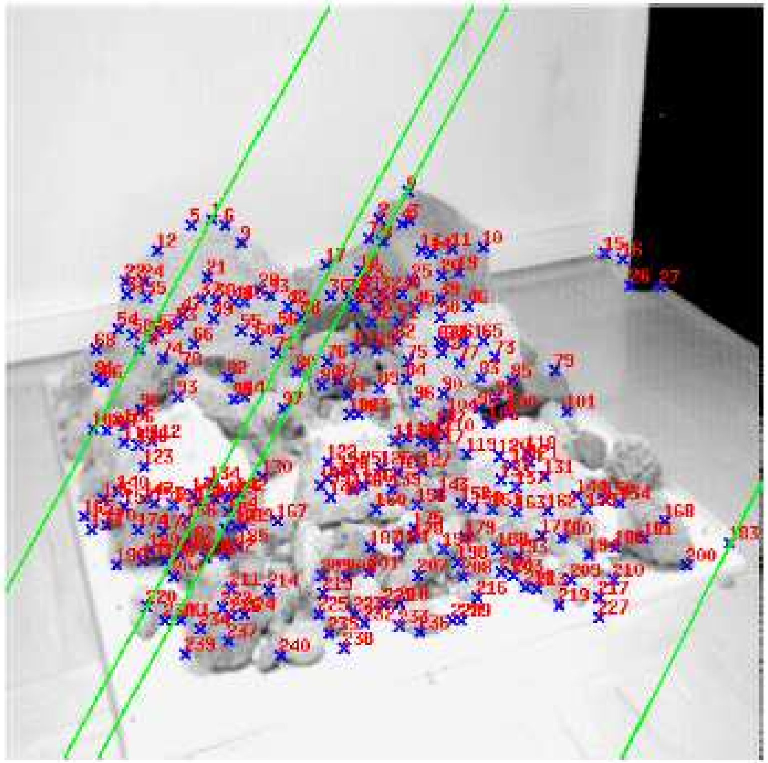
Fig. 4. Image pair used for comparing different estimation techniques of the fundamental matrix

Zhengyou Zhang
Determining the Epipolar Geometry and its Uncertainty: A Review
www.cs.unr.edu/~mircea/Courses/ cs790E/Lectures/zhang2.ppt

Example of Fundamental Matrix Estimation with Comparison

- The intrinsic parameters of both cameras and the displacement between them were computed offline through stereo calibration. The fundamental matrix computed from these parameters serves as a ground truth.
- There are 241 point matches, which are established automatically.

Example of Fundamental Matrix Estimation with Comparison



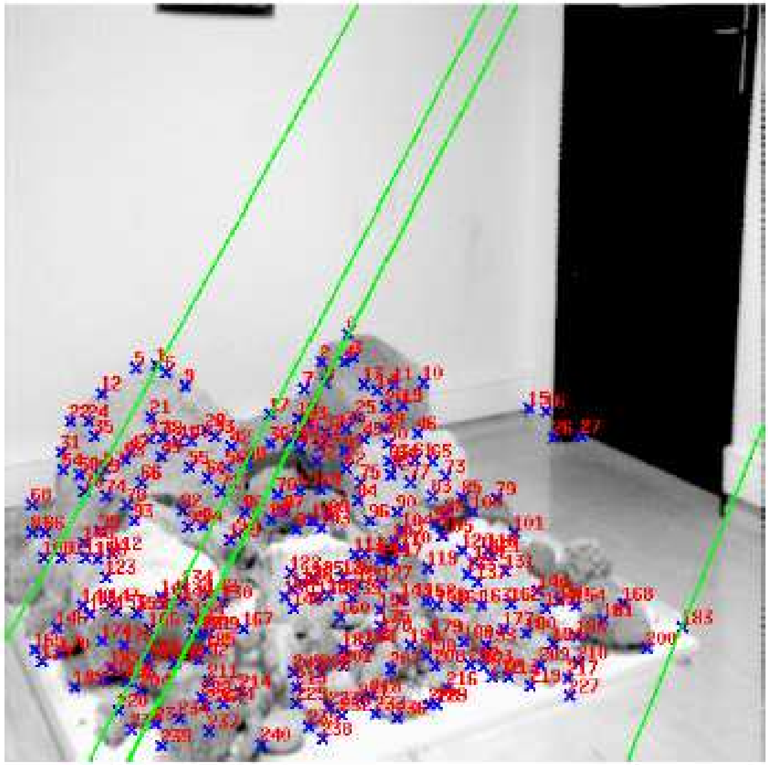


Fig. 5. Epipolar geometry estimated through classical stereo calibration, which serves as the ground truth

Zhengyou Zhang
Determining the Epipolar Geometry and its Uncertainty: A Review
www.cs.unr.edu/~mircea/Courses/ cs790E/Lectures/zhang2.ppt

Example of Fundamental Matrix Estimation with Comparison

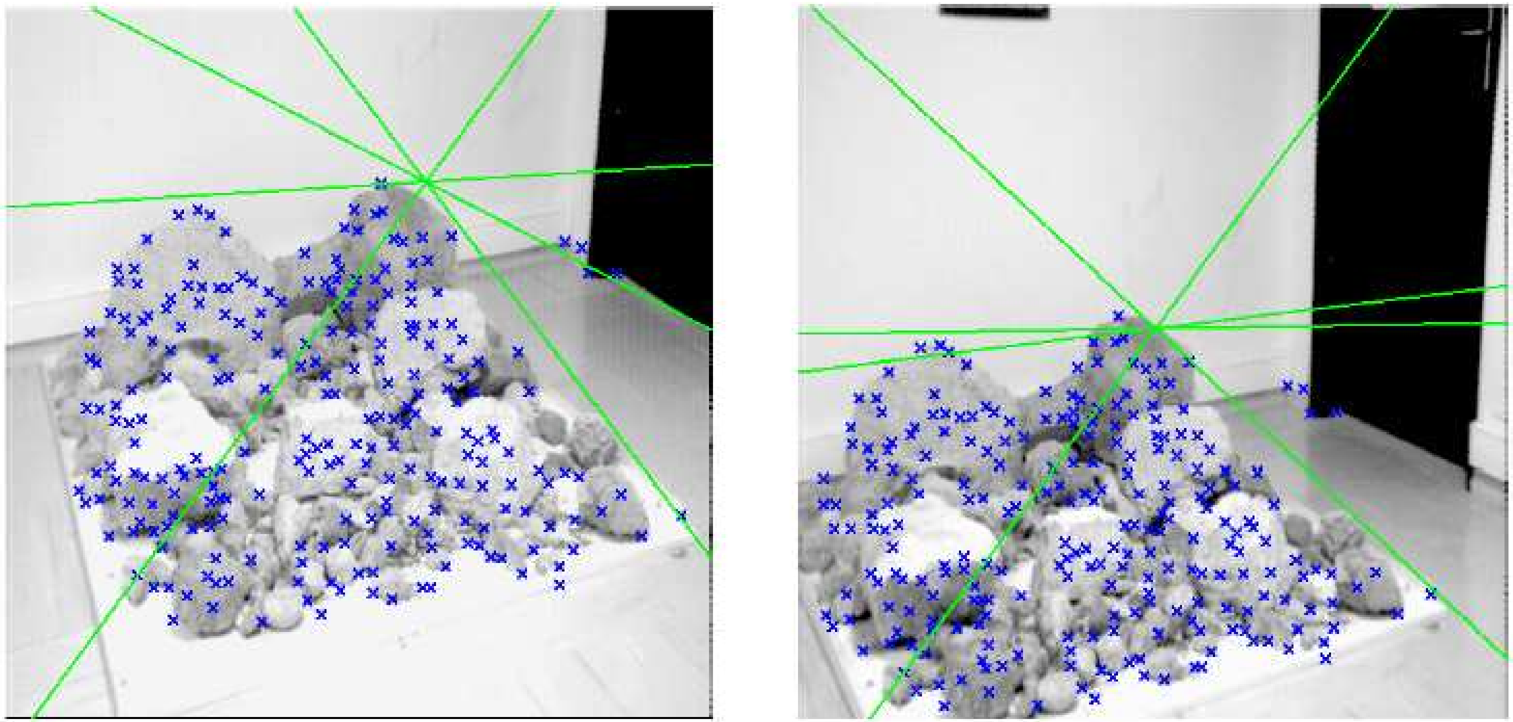


Fig. 6. Epipolar geometry estimated with the linear method

Zhengyou Zhang
Determining the Epipolar Geometry and its Uncertainty: A Review
www.cs.unr.edu/~mircea/Courses/ cs790E/Lectures/zhang2.ppt

Example of Fundamental Matrix Estimation with Comparison

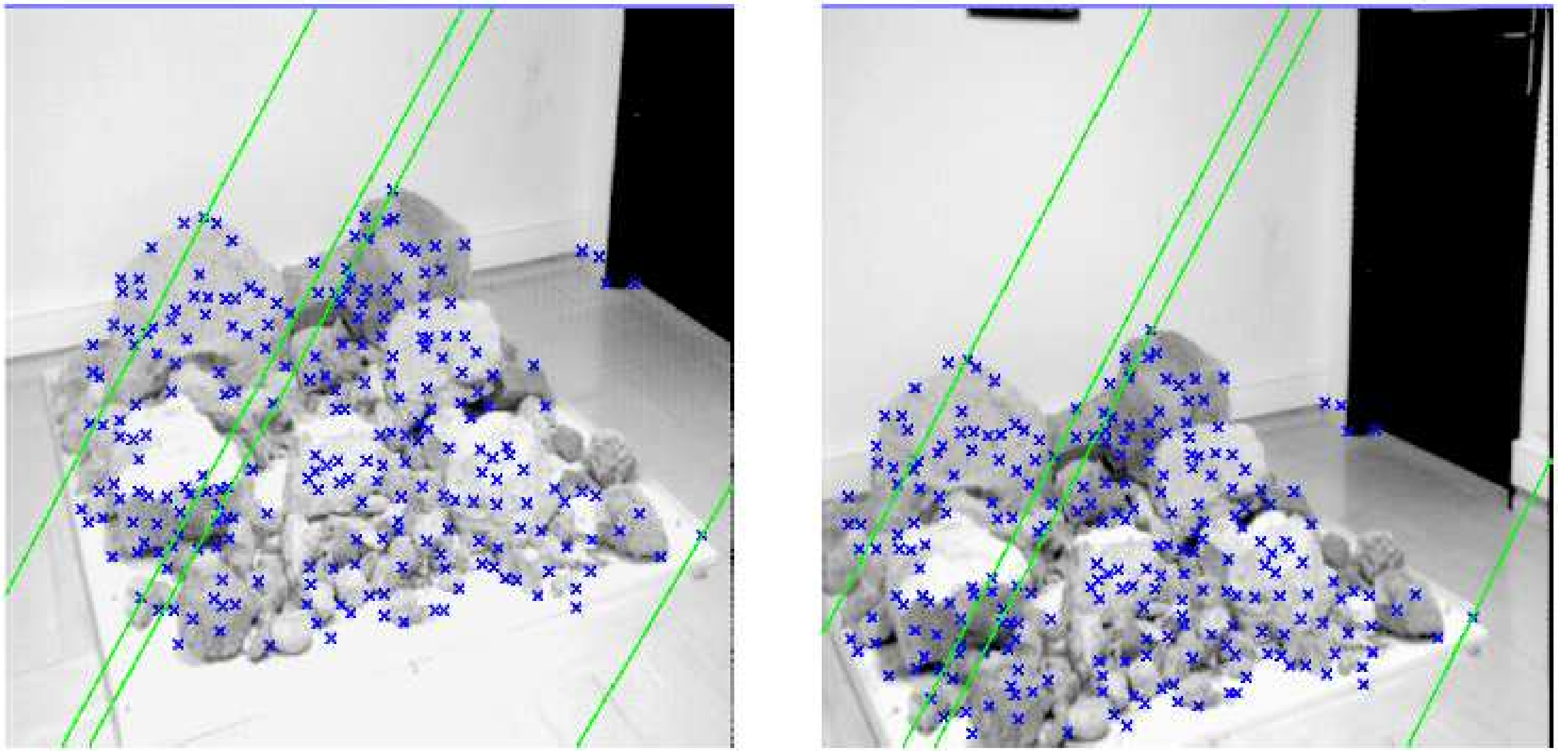


Fig. 7. Epipolar geometry estimated with the linear method with prior data normalization

Zhengyou Zhang
Determining the Epipolar Geometry and its Uncertainty: A Review
www.cs.unr.edu/~mircea/Courses/ cs790E/Lectures/zhang2.ppt

Example of Fundamental Matrix Estimation with Comparison

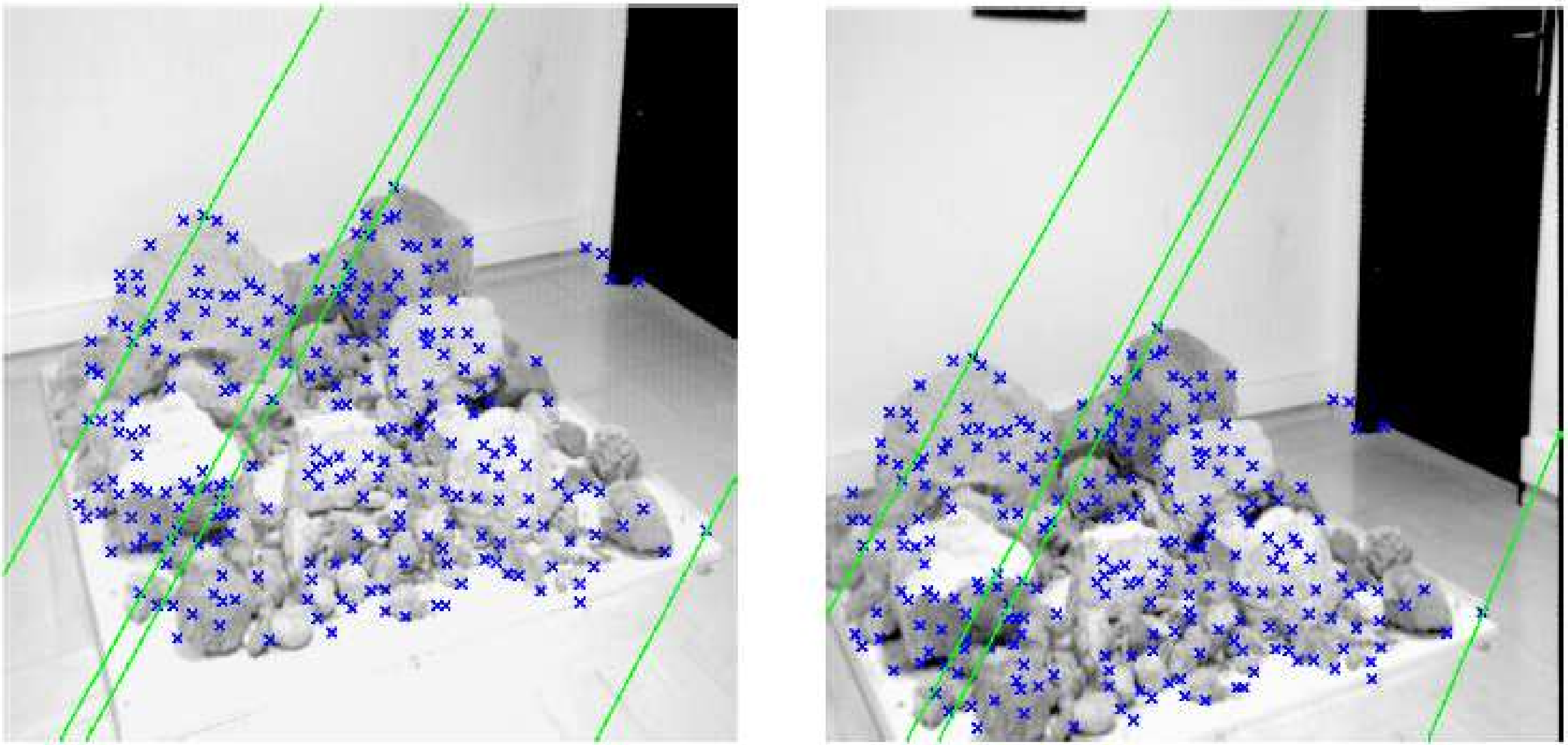


Fig. 8. Epipolar geometry estimated with the nonlinear method

Zhengyou Zhang
Determining the Epipolar Geometry and its Uncertainty: A Review
www.cs.unr.edu/~mircea/Courses/ cs790E/Lectures/zhang2.ppt

Example of Fundamental Matrix Estimation with Comparison

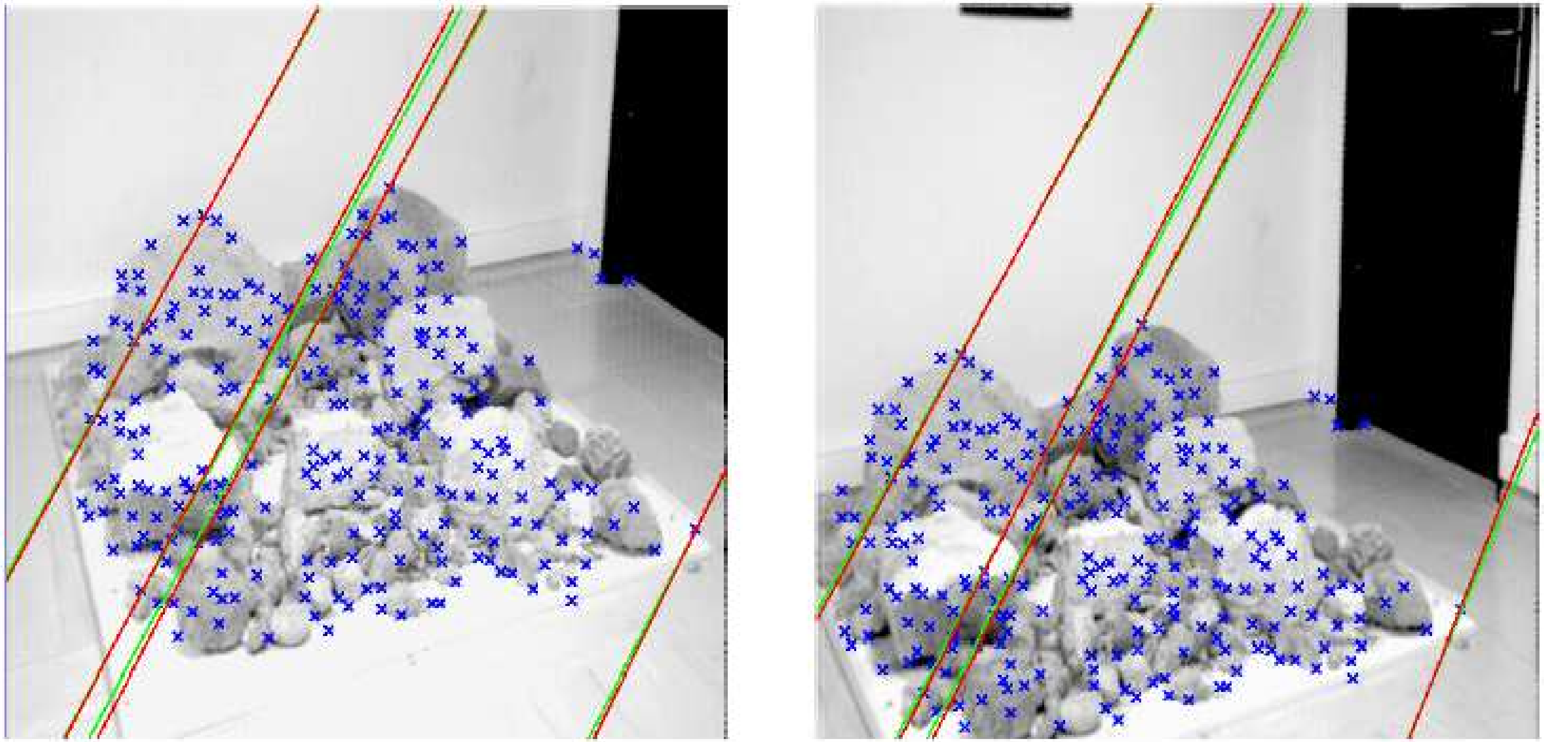


Fig. 9. Comparison between the Epipolar geometry estimated through classical stereo calibration (shown in Red/Dark lines) and that estimated with the nonlinear method (shown in Green/Grey lines)

Zhengyou Zhang
Determining the Epipolar Geometry and its Uncertainty: A Review
www.cs.unr.edu/~mircea/Courses/ cs790E/Lectures/zhang2.ppt

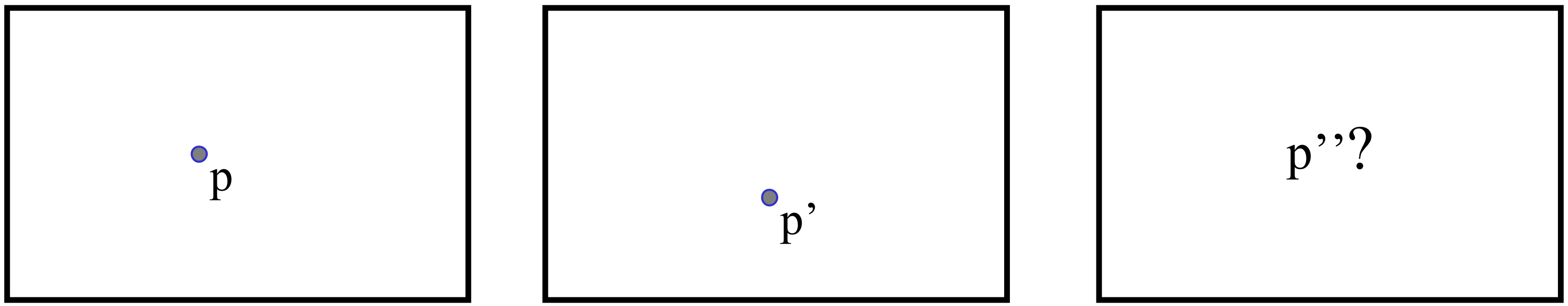
The fundamental matrix F

F is the unique 3x3 rank 2 matrix that satisfies $x'^T F x = 0$ for all $x \leftrightarrow x'$

(i) **Transpose:** if F is fundamental matrix for (P,P'), then F^T is fundamental matrix for (P',P)
(ii) **Epipolar lines:** $l' = Fx$ & $l = F^T x'$
(iii) **Epipoles:** on all epipolar lines, thus $e'^T F x = 0$, $\forall x$ $\Rightarrow e'^T F = 0$, similarly $Fe = 0$
(iv) **F** has 7 d.o.f. , i.e. 3x3-1(homogeneous)-1(rank2)
(v) **F** is a projective mapping from a point x to a line $l' = Fx$

Trinocular constraints

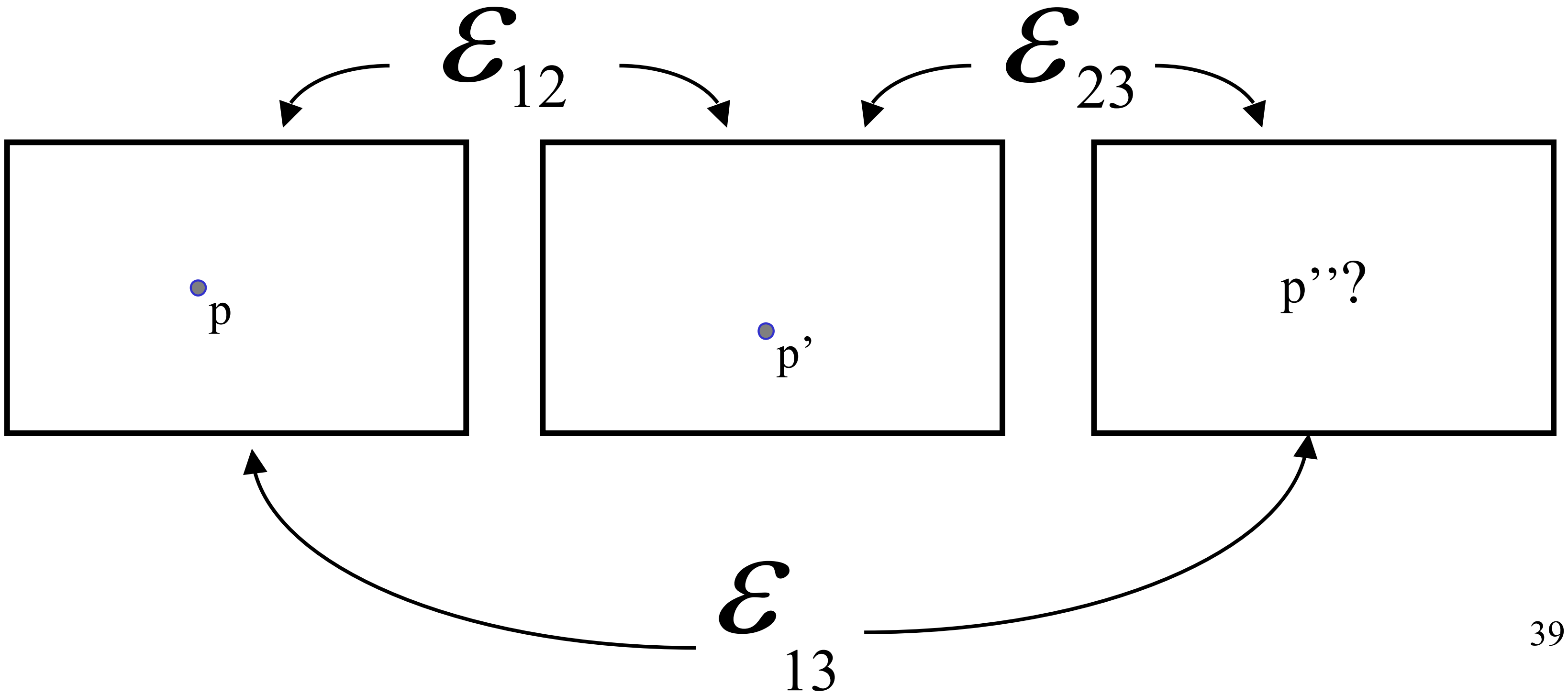
Given p’,p’’ in left and middle image, where is p’’ in a third view?



Three essential matrices

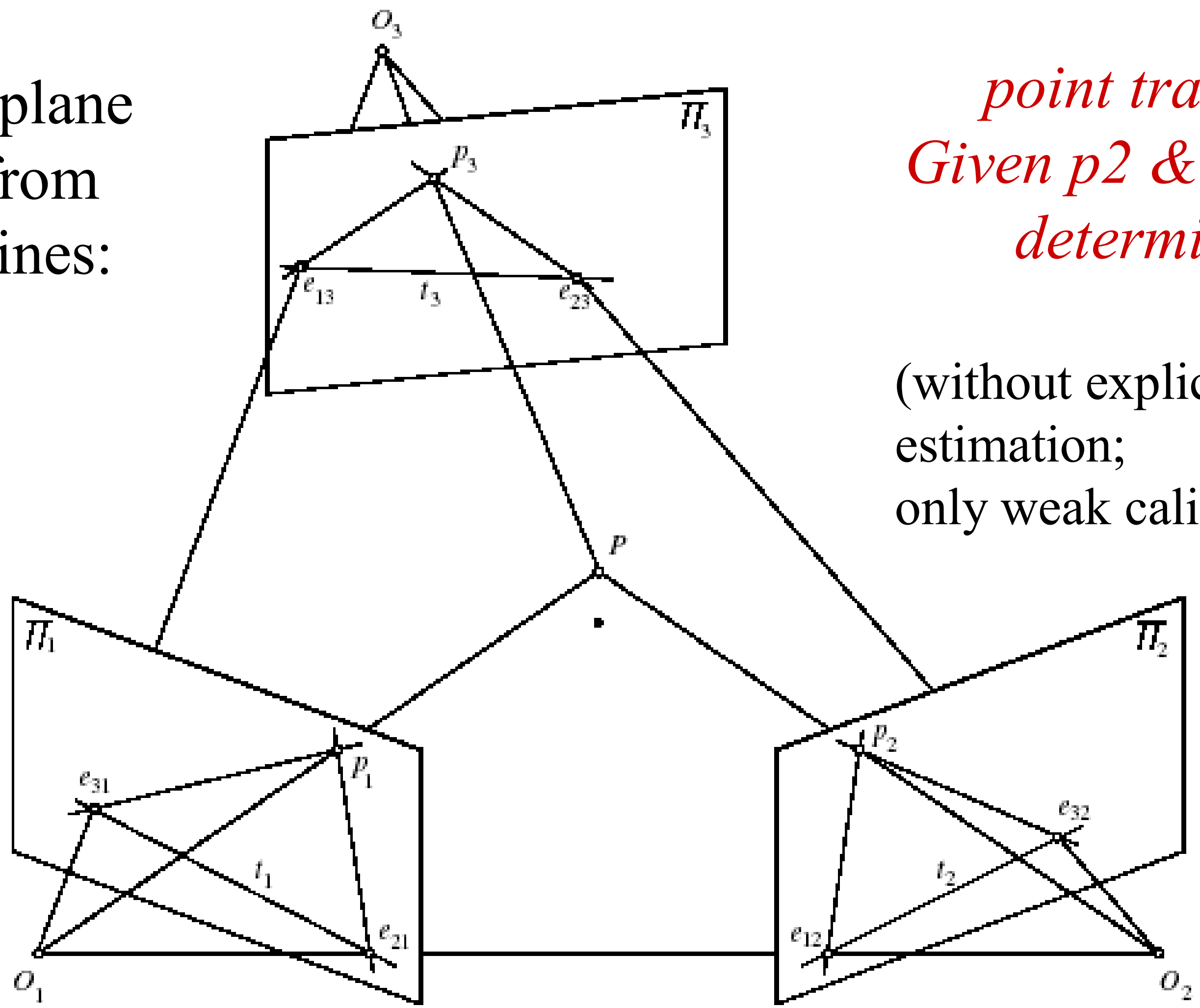
Essential matrices relate each pair:

(calibrated case)



Trinocular epipolar geometry

Trifocal plane formed from trifocal lines:



point transfer: Given p2 & p3, p1 is determined!

(without explicit depth estimation;
only weak calibration)

Three essential matrices

$$\begin{cases} \boldsymbol{p}_1^T \mathcal{E}_{12} \boldsymbol{p}_2 = 0, \\ \boldsymbol{p}_2^T \mathcal{E}_{23} \boldsymbol{p}_3 = 0, \\ \boldsymbol{p}_3^T \mathcal{E}_{31} \boldsymbol{p}_1 = 0, \end{cases}$$

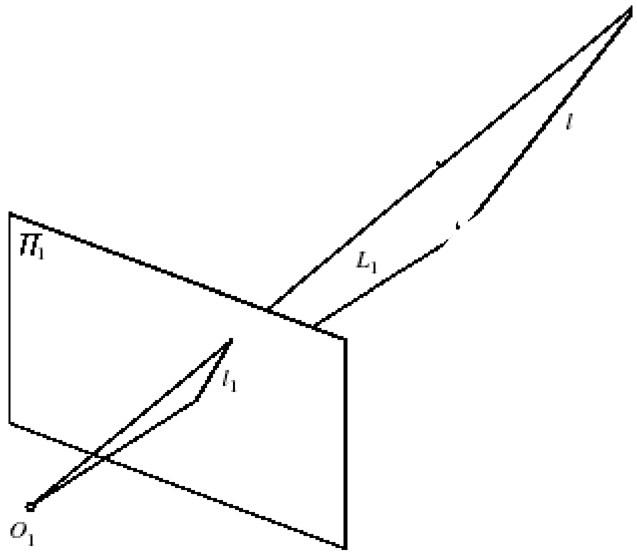
Any two are independent!

Can predict third point from two others.

Point transfer: e.g., solve for p_1 given p_2,p_3,$\boldsymbol{E}_{12}$,$\boldsymbol{E}_{31}$

Trifocal line constraint

Form the plane containing a line $\boldsymbol{l}$ and optical center of one camera:



$$\boldsymbol{l}^T \ \boldsymbol{p} = 0,$$

$$\boldsymbol{l}^T \mathcal{M} \boldsymbol{P} = 0,$$

Trifocal line constraint

3 cameras, 3 plane equations:

$$\begin{pmatrix} \boldsymbol{l}_1^T \mathcal{M}_1 \\ \boldsymbol{l}_2^T \mathcal{M}_2 \\ \boldsymbol{l}_3^T \mathcal{M}_3 \end{pmatrix} \boldsymbol{P} = \boldsymbol{0}$$

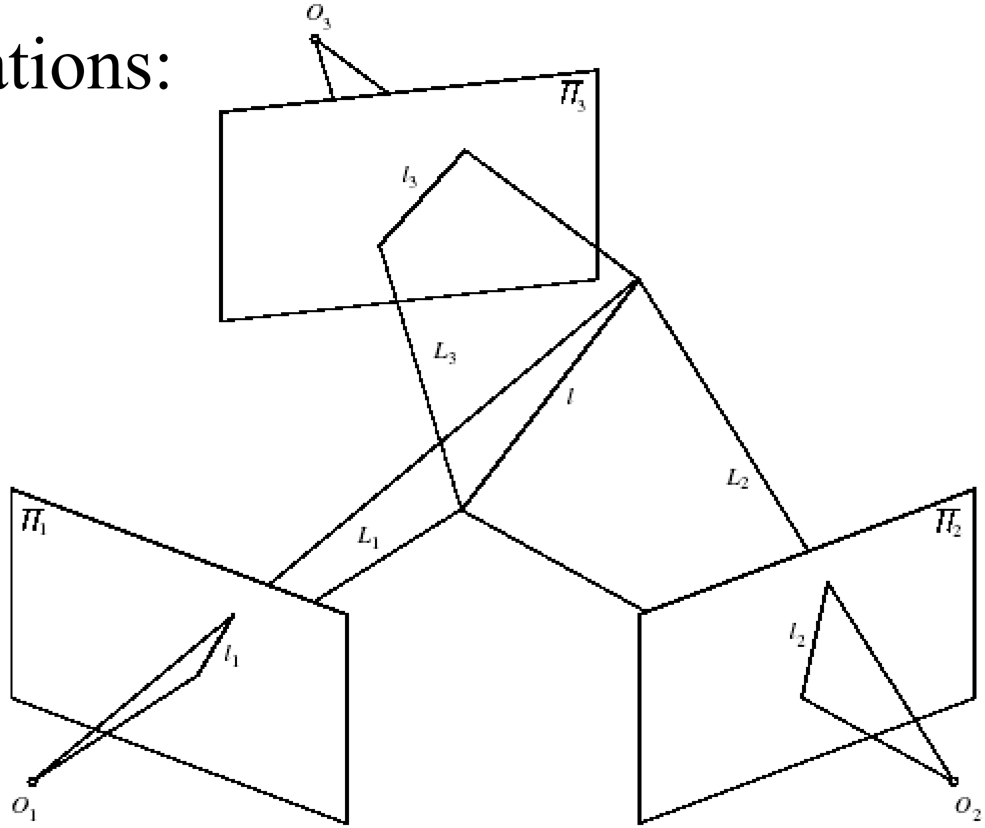


Figure 12.6. Three images of a line define it as the intersection of three planes in t same pencil.

$$\mathcal{L} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{l}_1^T \mathcal{M}_1 \\ \boldsymbol{l}_2^T \mathcal{M}_2 \\ \boldsymbol{l}_3^T \mathcal{M}_3 \end{pmatrix}$$

If 3 lines intersect in more than one point (a line) this system is degenerate and is rank 2.

Trifocal line constraint

Assume calibrated camera coordinates

$$\mathcal{M}_1 = \begin{pmatrix} \mathrm{Id} & \mathbf{0} \end{pmatrix}$$

$$\mathcal{M}_2 = \begin{pmatrix} \mathcal{R}_2 & \boldsymbol{t}_2 \end{pmatrix}$$

$$\mathcal{M}_3 = \begin{pmatrix} \mathcal{R}_3 & \boldsymbol{t}_3 \end{pmatrix}$$

then

$$\mathcal{L} = \begin{pmatrix} \boldsymbol{l}_1^T & 0 \\ \boldsymbol{l}_2^T \mathcal{R}_2 & \boldsymbol{l}_2^T \boldsymbol{t}_2 \\ \boldsymbol{l}_3^T \mathcal{R}_3 & \boldsymbol{l}_3^T \boldsymbol{t}_3 \end{pmatrix}$$

$$\mathcal{L} = \begin{pmatrix} \boldsymbol{l}_1^T & 0 \\ \boldsymbol{l}_2^T \mathcal{R}_2 & \boldsymbol{l}_2^T \boldsymbol{t}_2 \\ \boldsymbol{l}_3^T \mathcal{R}_3 & \boldsymbol{l}_3^T \boldsymbol{t}_3 \end{pmatrix}$$

Rank $\mathcal{L} = 2$ means det. of 3x3 minors are zero, and can be expressed as:

$$\boldsymbol{l}_1 \times \begin{pmatrix} \boldsymbol{l}_2^T \mathcal{G}_1^1 \boldsymbol{l}_3 \\ \boldsymbol{l}_2^T \mathcal{G}_1^2 \boldsymbol{l}_3 \\ \boldsymbol{l}_2^T \mathcal{G}_1^3 \boldsymbol{l}_3 \end{pmatrix} = \mathbf{0},$$

with

$$\mathcal{G}_1^i = \boldsymbol{t}_2 \boldsymbol{R}_3^{iT} - \boldsymbol{R}_2^i \boldsymbol{t}_3^T$$

The trifocal tensor

These 3 3x3 matrices are called the trifocal tensor.

$$\mathcal{G}_1^i = \boldsymbol{t}_2 \boldsymbol{R}_3^{iT} - \boldsymbol{R}_2^i \boldsymbol{t}_3^T$$

the constraint

$$\boldsymbol{l}_1 \times \begin{pmatrix} \boldsymbol{l}_2^T \mathcal{G}_1^1 \boldsymbol{l}_3 \\ \boldsymbol{l}_2^T \mathcal{G}_1^2 \boldsymbol{l}_3 \\ \boldsymbol{l}_2^T \mathcal{G}_1^3 \boldsymbol{l}_3 \end{pmatrix} = \boldsymbol{0},$$

can be used for point or line transfer.

Trifocal line constraint

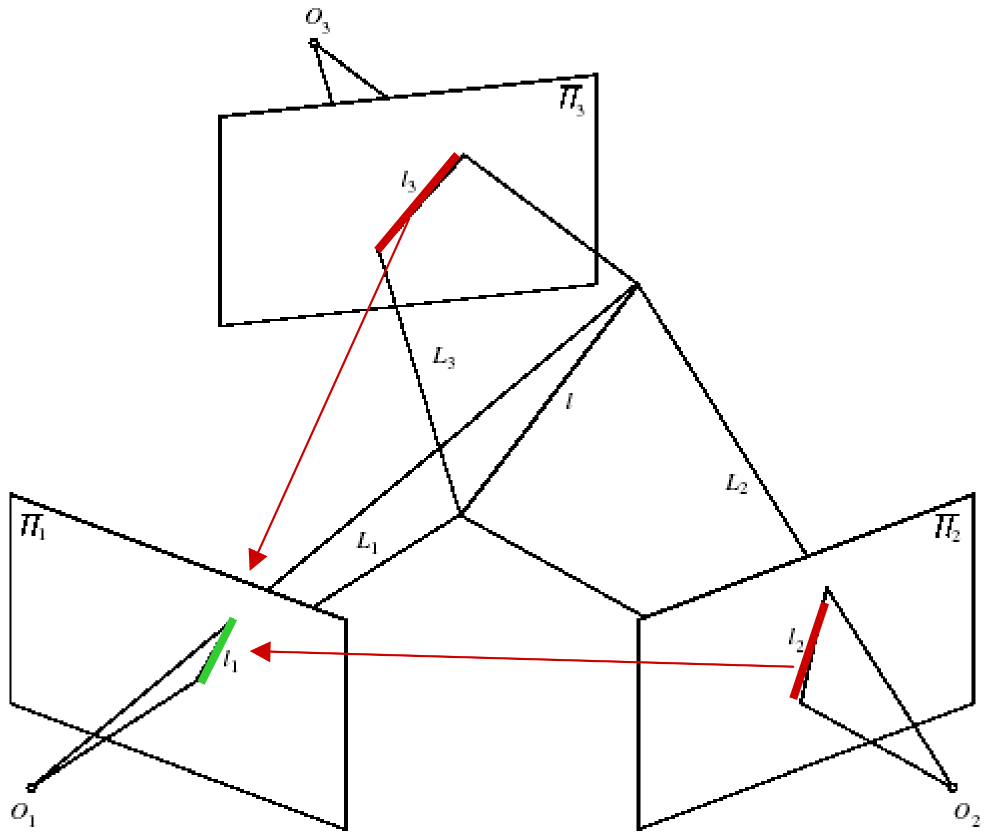
line transfer:

$$\boldsymbol{l}_1 \approx \begin{pmatrix} \boldsymbol{l}_2^T \mathcal{G}_1^1 \boldsymbol{l}_3 \\ \boldsymbol{l}_2^T \mathcal{G}_1^2 \boldsymbol{l}_3 \\ \boldsymbol{l}_2^T \mathcal{G}_1^3 \boldsymbol{l}_3 \end{pmatrix}$$

point transfer via lines: form independent pairs of lines through p2,p3, solve for p1.

Line transfer

$$\boldsymbol{l}_1 \approx \begin{pmatrix} \boldsymbol{l}_2^T \mathcal{G}_1^1 \boldsymbol{l}_3 \\ \boldsymbol{l}_2^T \mathcal{G}_1^2 \boldsymbol{l}_3 \\ \boldsymbol{l}_2^T \mathcal{G}_1^3 \boldsymbol{l}_3 \end{pmatrix}$$



Uncalibrated case

$$\mathcal{L} = \begin{pmatrix} \boldsymbol{l}_1^T \mathcal{K}_1 & 0 \\ \boldsymbol{l}_2^T \mathcal{K}_2 \mathcal{R}_2 & \boldsymbol{l}_2^T \mathcal{K}_2 \boldsymbol{t}_2 \\ \boldsymbol{l}_3^T \mathcal{K}_3 \mathcal{R}_3 & \boldsymbol{l}_3^T \mathcal{K}_3 \boldsymbol{t}_3 \end{pmatrix}$$

$$\mathcal{A}_i \stackrel{\text{def}}{=} \mathcal{K}_i \mathcal{R}_i \mathcal{K}_1^{-1} \qquad \boldsymbol{a}_i \stackrel{\text{def}}{=} \mathcal{K}_i \boldsymbol{t}_i$$

$$\mathcal{M}_1 = (\, \mathcal{K}_1 \quad \mathbf{0} \,),\ \mathcal{M}_2 = (\, \mathcal{A}_2 \mathcal{K}_1 \quad \boldsymbol{a}_2 \,),$$

$$\mathcal{M}_3 = (\, \mathcal{A}_3 \mathcal{K}_1 \quad \boldsymbol{a}_3 \,)$$

$$\mathrm{Rank}(\mathcal{L}) = 2 \Longleftrightarrow \mathrm{Rank}(\mathcal{L} \begin{pmatrix} \mathcal{K}_1^{-1} & 0 \\ 0 & 1 \end{pmatrix}) = \mathrm{Rank} \begin{pmatrix} \boldsymbol{l}_1^T & 0 \\ \boldsymbol{l}_2^T \mathcal{A}_2 & \boldsymbol{l}_2^T \boldsymbol{a}_2 \\ \boldsymbol{l}_3^T \mathcal{A}_3 & \boldsymbol{l}_3^T \boldsymbol{a}_3 \end{pmatrix} = 2$$

Project

The final project may be

- An original implementation of a new or published idea
- A detailed empirical evaluation of an existing implementation of one or more methods
- A paper comparing three or more papers not covered in class, or surveying recent literature in a particular area

A project proposal not longer than two pages must be submitted and approved by April 1st.

Project

March 16: Project previews / Brainstorming

3-5 minute presentation of

- Specific Project idea
- Your recent research, or thesis proposal (if it relates to vision)
- Paper you are interested in and think may form the basis of a project
- Area you wish to write a survey paper on; list major papers…