Quantum mechanics on space-time lattices using path integrals in a Minkowski metric

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Discrete Representations of n-Dimensional Wave Equations and Their Applications to Quantum Mechanics

by

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Submitted to the Department of Physics
in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

at the

Massachusetts Institute of Technology

July, 1992

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July 5, 1992

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Abstract

A simple model of many-body quantum mechanics governed by completely local interactions in ordinary three-dimensional space is presented. The equations of motion, involving no Euclideanization of time, are elementary extensions of those describing random walks on a square lattice, and are far simpler than those obtained through standard path-integral theory. A quantity is defined in terms of populations of particles on the lattice, whose expectation value is proportional to the square of the multi-particle wave functions associated with the system; for sufficiently microscopic measurement scales, this generalized 'density' may take on negative values, thereby circumventing the constraints imposed by Bell's inequality.

1 Introduction

The general solution of the one-dimensional wave equation may be written as a sum of two functions, or traveling waves. The equations of motion of these traveling waves, first-order in time and space, are simple translations in opposite directions with a constant speed.

Section 2 of this paper extends the benefits of such an approach to higher dimensions, so that in general, the n-dimensional wave equation is likewise recast as a sum of $2n$ functions, first order in time, each of which is associated with motion along the positive or negative directions of the n axes of the space. The domain of the wave functions is a discrete space-time lattice; nevertheless, the resultant wave equations and their solutions display relativistic covariance in the continuum limit.

Section 3 shows that this traveling wave decomposition lends itself to a statistical implementation. Recall that the kernel of the diffusion equation is expressible as the continuum limit of distributions of random walks on a lattice, and that the time evolution of the diffusion equation may be computed by ensembles of particles executing such random walks. By assigning these particles a (discrete) phase factor, it is possible to give the wave equation a similar implementation, one that is much simpler than those obtained by way of standard path integral theory.

Section 4 shows that this statistical rendering of the wave equation can be used to construct a model of multi-particle quantum mechanics, in an ordinary three-dimensional configuration space, in a Minkowski metric. In general, a single particle state (or collection thereof) is represented, or simulated, by a gas-like distribution of particles propagating throughout the lattice at once. A large ensemble of such distributions may then be used to obtain the correct quantum mechanical probabilities associated with both single and multi-particle states. This is because there exists a quantity, additive in the number of these ensemble members, whose expectation value is likewise equal to the square of the corresponding wave functions.

Section 5 deals with issues that must be considered in applying the quantum mechanical aspects of this formalism to macroscopic phenomena. The final section discusses related formalisms, and compares them to the present one. The appendices show how the traveling wave decomposition may be applied to the scalar Klein-Gordon equation, the Dirac equation, and Maxwell's equations in a Lorentz gauge. Each of these cases are straightforward generalizations of the scalar equations, whereby the topology of the lattice and the coefficients of the associated transition matrices are altered in direct correspondence to the internal structure of the associated particles. The mass, and also the potentials, are then introduced as perturbations induced by Poissonly distributed random fields, bypassing the more complicated formalism of Higgs particles. The appendix also considers some mathematical properties of the associated scalar wave solutions.

2 The Discrete Wave Equation

The discrete analogue of the wave equation is defined on an orthonormal, $(n+1)$-dimensional space-time lattice of points whose spacing is unity. The second order partial derivatives of time and space appearing in the D'Alembertian are replaced by their usual finite-difference analogues

$$\frac{\partial^2}{\partial x_i^2}\psi(\ldots,x_i,\ldots) \equiv \psi(\ldots,x_i+1,\ldots) - 2\psi(\ldots,x_i,\ldots) + \psi(\ldots,x_i-1,\ldots) \tag{1}$$

where

$$x_i = x_1, x_2, ..., x_n, t \quad .$$

Thus, the wave equation retains its usual form,

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)\psi \ . \tag{2}$$

Setting the constant c, which will be called the speed of light, to $\sqrt{1/n}$ confers several useful properties on the equations, among them, a conserved momentum and energy which in the continuum limit converge to their standard forms [1]; this is done throughout. In particular, because of the cancellation of all terms in the above equation corresponding to the central term of the right-hand side of (1), the space-time is decoupled into two distinct lattices, even and odd (Figure 1), such that every space-time point $(\mathbf{x}, t)$ in the even (odd) sublattice has the property that

$$t + \sum_{i=1}^{n} x_i \tag{3}$$

is even (odd). Without loss of generality, it will be assumed that the solutions considered here are nonzero on only one of these lattices, unless stated otherwise.

It can be seen from the two preceding equations that solutions of the discrete wave equations form a vector space whose dimensionality is equal to twice the number of points in the lattice (i.e., the sublattice). An element of this space of solutions is completely determined by arbitrarily assigning values on all points of the lattice at two successive times.

Any solution of the n-dimensional discrete wave equation defined on an n-cubical lattice of N^n points may also be given in terms of harmonic solutions via Fourier analysis, e.g.,

$$\psi(\mathbf{x}, t) = \prod_{j=1}^{n}\sum_{\kappa_j=0}^{N-1}\mathcal{A}(\vec{\kappa})e^{\frac{2\pi i}{N}(\vec{\kappa}\cdot x+\omega t)} \ , \tag{4}$$

where the $\mathcal{A}(\vec{\kappa})$ depend on the initial conditions. The frequency ω, whose functional dependence on the components of $\vec{\kappa}$ has been suppressed for clarity, is given by the

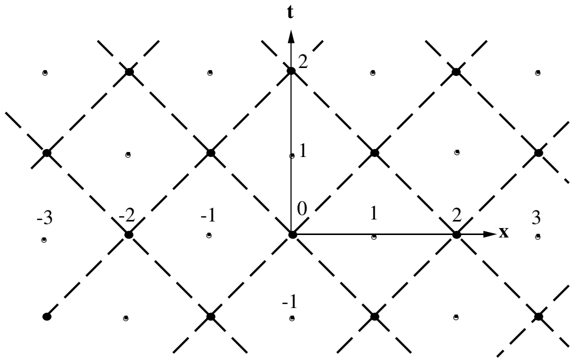

Figure 1: Two-dimensional discrete space-time is decoupled into two distinct lattices for a certain value of the speed of light; the diagonal arcs shown connect the nearest-neighbor points of one such lattice. Higher dimensional space-times are likewise decoupled into two lattices.

dispersion relation

$$\sin^2 \frac{\omega}{2} = \frac{1}{n} \left(\sin^2 \frac{\kappa_1}{2} + \cdots + \sin^2 \frac{\kappa_n}{2} \right) \ . \tag{5}$$

This passes in the continuum limit to the familiar

$$\omega^2 = \frac{1}{n} \left(k_1^2 + \cdots + k_n^2 \right) \ , \tag{6}$$

where the k_x and k_y are the respective conjugate variables of the Fourier integral appearing in the continuum generalization of (4). Note that if $c^2 = 1/n$ as implied above, the numerical integration of (2) is stable. If one were to make the a priori simpler choice of setting $c^2 = 1$, or indeed to any value greater than $\sqrt{1/n}$, then for all except the one-dimensional case, there would be spatial frequencies for which the modulus of the right-hand side of (5) would be greater than one. The time frequency ω would then have to contain an imaginary component in order for the equality to hold, leading in general to an exponential growth of the solutions. As it is, the solutions retain the characteristic undamped periodicity of their continuum analogues. Moreover, their deviation from the continuum case stays bounded. Of course, only solutions whose associated wavelengths are much larger than the lattice spacing can be well approximated by such discrete analogues, but in principle the correspondence may be made as close as desired.

2.1 The one-dimensional case

Traveling waves and arcs

As is well known, the general solution of the one-dimensional discrete wave equation may be written as the sum of two traveling waves,

$$\psi(x,t) = f_+(u) + f_-(v) \ , \tag{7}$$

where these two traveling waves are respectively functions of the single variables $u = x - t$ and $v = x + t$, so that they translate unchanged in opposite directions as time passes.

Indeed, this propagation from one space-time point to its neighbors suggests that these traveling waves are functions more naturally defined on the *arcs* that may be said to connect neighboring space-time points (cf. Figure 1); however, the utility of such a view becomes fully manifest only in higher dimensions. Let $f_+^{\rm in}(x,t)$ and $f_-^{\rm in}(x,t)$ denote functions, which will be referred to as *flows*, defined on arcs connecting space-time points of the form $(x \mp 1, t-1)$. At the risk of redundancy, let $f_\pm^{\rm out}(x,t)$ similarly denote functions defined on arcs connecting space-time points of the form $(x \pm 1, t+1)$. Obviously,

$$f_\pm^{\rm in}(x,t) \equiv f_\pm^{\rm out}(x \mp 1, t-1) \ . \tag{8}$$

The decomposition of (7) then implies

$$f_\pm^{\rm in}(x,t) = f_\pm^{\rm out}(x,t) \ . \tag{9}$$

Any solution of (2) may then be expressed in terms of functions defined on the arcs.

Like the set of point solutions, the set of arc solutions also forms a vector space, and the traveling wave decomposition implies that there exists a homomorphism from the arc solutions to the point solutions (Figure 2). Moreover, this homomorphism is preserved over time, under the respective time evolution of each system. By the implicit use of this homomorphism, (7) may be restated as

$$\psi(x,t) = f_+^{\rm in}(x,t) + f_-^{\rm in}(x,t) = f_+^{\rm out}(x,t) + f_-^{\rm out}(x,t) \ . \tag{10}$$

The homomorphism is obviously not one-to-one, since by everywhere setting

$$f_+^{\rm out}(x,t) = -f_-^{\rm out}(x,t) = \alpha \ ,$$

where α is an arbitrary constant, one obtains an arc solution that under the homomorphism is mapped onto the trivial (i.e., everywhere vanishing) point solution. However, by a proper choice of the basis functions, the expression of a given solution in terms of the flows can (in any number of dimensions) be made unique, so that the relation between the point solutions and arc solutions becomes an isomorphism.

$$
\begin{array}{ccc}
\{\sum f(x,t)\} & \xrightarrow{(9)} & \{\sum f(x,t+1)\} \\
| & & | \\
\mathcal{H} & & \mathcal{H} \\
\downarrow & & \downarrow \\
\{\psi(x,t)\} & \xrightarrow{(2)} & \{\psi(x,t+1)\}
\end{array}
$$

Figure 2: There exists a homomorphism $\mathcal{H}$ from the space of arc solutions to the space of point solutions that is preserved under their respective time evolutions.

Completeness

In considering the notion of completeness, for any number of dimensions, the space in question is assumed for convenience to be a torus of length $2N$ in all spatial dimensions, where N is furthermore an even number. Moreover, let all solutions have the property that

$$\psi(\ldots, x_i, \ldots) = -\psi(\ldots, x_i + N, \ldots) \ , \tag{11}$$

where the '+' sign in the expression $x_i + N$ implies addition modulo N. Therefore, a solution is completely determined by its boundary conditions on the subspace of points for which the x_i range from 0 to $N-1$. By making N as large as necessary, the desired generality is retained. (Choosing N to be even makes the initialization described below identical, up to a change of sign, for each subspace. Imposing the parity condition (11) excludes the unphysical or uninteresting solutions containing zero frequency modes or terms linear in the time or space variables.)

Returning again to one-dimensional systems, let $h_{\pm}(x-x_0)$ stand for the traveling wave solutions that at $t = t_0$ are zero everywhere except on the arcs going out from the point x_0 to the respective point $x_0 \pm 1$, where they are 1. Such solutions, and their n-dimensional generalizations, will be called *hodotic* solutions (from the Greek word for "path"); they have many unusual properties, especially in higher dimensional spaces, some of which will be discussed in the appendix. For purposes of clarity, their time dependence has been suppressed.

A moment's thought will show that under the homomorphism, these arc solutions correspond to the point solutions that at $t = 0$ are equal to δ_{x,x_0} and at $t = 1$ are respectively equal to $\delta_{x,x_0\pm 1}$, where $\delta_{x,x'}$ is the Kronecker delta.

Next, let $G_0(x-x_0)$ and $G_1(x-x_1)$, be the point solutions of the wave equation whose initial conditions (at times t_0 and $t_1 = t_0 + 1$), are

$$
\begin{aligned}
G_0(x-x_0) &= \delta_{x,x_0}\delta_{t,t_0} \\
G_1(x-x_1) &= \delta_{x,x_1}\delta_{t,t_1}
\end{aligned}
\tag{12}
$$

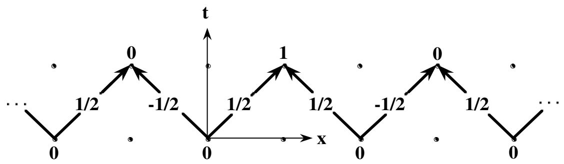

Figure 3: An arrangement of hodotic solutions that yields a Green's function $G_1(x - x_1)$ for the discrete wave equation.

$$x, x_0, x_1 \in \{0, 1, \ldots N-1\} \ .$$

Again, their dependence on time has been suppressed.

It is clear that any solution of the wave equation may be expressed as a linear combination of the G_0 and G_1. On the subspace of length N mentioned above, The function $G_0(x - x_0)$ and $G_1(x - x_1)$ may be expressed in terms of the arc functions h_+ and h_- as

$$\begin{aligned} G_0(x - x_0) = \frac{1}{2} \sum_{x'=0,2,\ldots}^{N-2} & \{h_+(x - x_0 - x')\mathrm{Sgn}(x - x_0) \\ & - h_-(x - x_0 - x')\mathrm{Sgn}(x - x_0 - 1)\} \qquad (13a) \end{aligned}$$

and

$$\begin{aligned} G_1(x - x_1) = \frac{1}{2} \sum_{x'=1,3,\ldots}^{N-1} & \{-h_+(x - x_1 - x')\mathrm{Sgn}(x' - x_1) \\ & + h_-(x - x_1 - x')\mathrm{Sgn}(x' - x_1)\} \ , \qquad (13b) \end{aligned}$$

where

$$\mathrm{Sgn}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{otherwise.} \end{cases}$$

Again, note that if t_0 is even (odd), then the solutions in question are assumed to be nonzero on the even (odd) sublattice; likewise, it is assumed that x_0 is even (odd), while x_1 is odd (even).

As the equations show, these Green's functions for the discrete wave function may be constructed by interlocking positive and negative hodotic solutions of opposite sign (Figure 3). Therefore, the traveling waves completely suffice to specify the solutions of the wave equation.

Note that these Green's functions require nonzero flows across the entire space. Therefore, even if the solution one wishes to simulate is initially nonzero only in some localized proper subset of the space—say, one that can be covered by a line segment (or in n dimensions, an n-cube)—its expression in terms of the Green's functions involves nonzero flows in a region extending across the entire space. However, if this (point) solution satisfies the additional property that its discrete integral over space is constant (as (2) implies it would be if it is the same at any two successive times), then one can find an arrangement of hodotic solutions that is also nonzero only in a localized subset.

2.2 The two-dimensional case

In passing to the two-dimensional case via the present analysis, solutions of the wave equation are again to be expressed as a linear sum of components. Each of these components will likewise be associated with flows along the arcs connecting nearest neighbors. However, since in higher dimensions even a localized wave packet spreads and deforms, it is to be expected that there will be mixing among these modes of propagation, instead of the trivial translation found in the one-dimensional case.

At any time, let each lattice point be viewed as a kind of black box, into which enter and from which exit four amplitudes. The latter are to be determined solely in terms of the former, in a linear fashion. If, as in the one-dimensional case, the flow out from the point (x, y, t) and in to the point $(x+1, y, t+1)$ is denoted as $f^{\text{out}}_{x+}(x, y, t)$, or equivalently as $f^{\text{in}}_{x+}(x+1, y, t+1)$, and the flows to other points are analogously denoted, these considerations may be restated as

$$\psi(x, y, t) = \sum_{\sigma} f^{\text{in}}_{\sigma}(x, y, t) = \sum_{\sigma} f^{\text{out}}_{\sigma}(x, y, t) \ , \tag{14}$$

and

$$f^{\text{out}}_{\sigma} = \sum_{\sigma'} c^{\text{in}}_{\sigma,\sigma'} f^{\text{in}}_{\sigma'} \ , \tag{15}$$

where $\sigma, \sigma' \in \{x_+, x_-, y_+, y_-\}$, and where the coefficients of the matrix $c^{\text{in}}_{\sigma,\sigma'}$ are to be determined.

Consider the solution of the wave equation corresponding to a nonzero flow in only one arc, say, $f^{\text{in}}_{x+}(0,0,0) = 1$, with all other incoming flows to all other points being zero; i.e., a two-dimensional hodotic solution. As in the one-dimensional case, the boundary conditions for the corresponding point solution are are $\psi(x, y, 0) = \delta_{x,0}\delta_{y,0}$, and $\psi(x, y, 1) = \delta_{x,1}\delta_{y,0}$.

Since the wave amplitude is thus specified on the entire space at two successive times, one can iterate these boundary conditions according to the wave equation (Figure 4), and thus uniquely determine the first column of the matrix $c^{\text{in}}_{\sigma,\sigma'}$. Repeating this analysis for the hodotic solutions of the remaining three directions completes the specification of the matrix, showing that

$$\begin{pmatrix} f_{x+} \\ f_{x-} \\ f_{y+} \\ f_{y-} \end{pmatrix}^{\text{out}} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{pmatrix} f_{x+} \\ f_{x-} \\ f_{y+} \\ f_{y-} \end{pmatrix}^{\text{in}} \ . \tag{16}$$

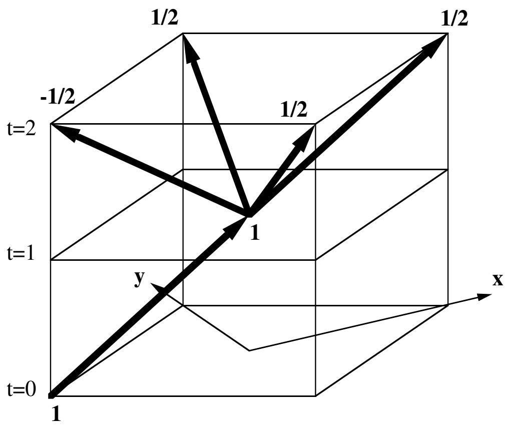

Figure 4: The x_+ hodotic solution, at $t = 0, 1$ and 2.

By using the time invariance properties of the wave equation, one also could have just as easily determined the coefficients of the inverse of the above matrix, $c^{\text{out}}_{\sigma,\sigma'}$, and would have found them to be identical. It is then a matter of algebra to show that the sum of the ingoing or outgoing flows at any point does indeed obey the discrete wave equation.

Note that the sum of the coefficients along any column of the matrix is one, so that the sum of flows entering a point at any time is equal to the sum of the exiting flows. This is also obvious from (14). Furthermore, the matrix is unitary, so that the sum of the squares of the entering (and thus the exiting) flows is conserved. Thus the evolution of such a system can be viewed as a network flow of a conserved quantity.

2.3 Higher dimensional cases

In higher dimensions, the hodotic solutions $h_{i_\pm}(\mathbf{x} - \mathbf{x}_0, t)$ are defined (in terms of their point values on the lattice) by the initial conditions

$$h_{i_\pm}(\mathbf{x} - \mathbf{x}_0, t) = \begin{cases} \delta_{\mathbf{x},\mathbf{x}_0} & \text{for } t = t_0 \\ \delta_{\mathbf{x},\mathbf{x}_0 \pm \mathbf{e}_i} & \text{for } t = t_0 + 1 \end{cases}, \tag{17}$$

where the j^{th} component of $\mathbf{e}_i$ has the value δ_{ij}. In terms of their arc amplitudes, the hodotic solutions are initially equal to unity on the $x_{i_\pm}$ arc leading out of the point $\mathbf{x}_0$, and zero on all the others.

The suppositions (14) and (15), and subsequently utilizing the associated hodotic solutions in order to determine the coefficients of the transition matrix, can be readily

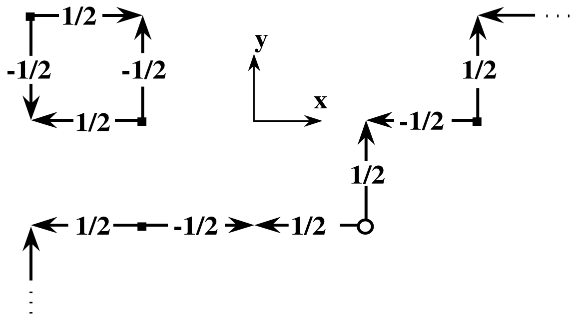

Figure 5: A two-dimensional non-collinear arrangement of hodotic solutions that yields a Green's function for the discrete wave equation. Nearest-neighbor points are displaced by one unit of time. The circle marks the support of the point solution.

generalized to higher dimensions. For example, in three dimensions, the transition matrix is determined to be

$$\begin{pmatrix} f_{x+} \\ f_{x-} \\ f_{y+} \\ f_{y-} \\ f_{z+} \\ f_{z-} \end{pmatrix}^{\text{out}} = \frac{1}{3} \begin{bmatrix} 1 & -2 & 1 & 1 & 1 & 1 \\ -2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -2 \\ 1 & 1 & 1 & 1 & -2 & 1 \end{bmatrix} \begin{pmatrix} f_{x+} \\ f_{x-} \\ f_{y+} \\ f_{y-} \\ f_{z+} \\ f_{z-} \end{pmatrix}^{\text{in}} . \tag{18}$$

In the general n-dimensional case, the coefficients of the matrix in any row or column will be $1/n$, except for the coefficients connecting oppositely directed flows, (e.g., $c_{x+,x-}$), which will be $-(n-1)/n$.

In each case, the number of traveling waves is determined by the number of adjacent neighbors of the points of the lattice, and the evolution of a given wave may be determined by the study of the corresponding hodotic solutions.

The considerations of completeness may be directly generalized to higher dimensions merely by imbedding the one-dimensional expressions for the Green's functions (13) along any of the axes. This is arguably the simplest way to express the Green's functions, although in higher dimensions more elaborate expressions are possible (Figure 5), since the overcompleteness of the hodotic solutions is more extensive.

It is possible to formally extend the notions of traveling waves to non-orthonormal lattices, generalizing the discrete wave equation to hexagonal, tetragonal, or indeed to any set of points where each point j has some privileged subset $\mathcal{N}(j)$ of generalized nearest neighbors; the "wave equation" then becomes

$$\psi(j, t+1) + \psi(j, t-1) = \frac{2}{|\mathcal{N}(j)|} \sum_{j' \in N(j)} \psi(j', t) \ , \tag{19}$$

where $|\mathcal{N}(j)|$ is the number of nearest neighbors—not necessarily constant throughout the space—corresponding to the point j. Again, each pair of neighbors will induce a flow, with an interaction among flows that may be determined by studying the behavior of the associated hodotic solutions. As before, there will be a conservation of the sum of the flows, and also of the sum of their squares, since the transition matrix is in all cases unitary. Of course, how well, if at all, the resultant system mimics the continuum wave equation in some limit depends on the particular lattice employed and, if definable, the associated dispersion relation. S. Gudder has studied even more general systems of flows on lattices and graphs, subject only to the condition that the transition matrix at each lattice point or graph node satisfies a unitarity condition[2].

Effective dimensionality

As an aside, note that the discreteness of lattice systems allows the number of dimensions to be easily manipulated in order to apply the above results to other systems of equations. Consider an orthonormal toroidal lattice, as before, and suppose the number of dimensions n, is very large, but that all except m dimensions are exactly one unit long. This means that for all but the first m dimensions, the arcs and flows leading out of a point will lead back in to the very same point. That is, the origin and terminus of such an arc are one and the same. This also means that the even and odd parity sublattices are now one and the same.

In effect, such a system is m-dimensional; however, the speed of light is still $\sqrt{n}$ rather than $\sqrt{m}$, and the number of types of flows is likewise equal to twice the actual dimension n, rather than m.

The utilization of such systems with "thin" dimensions is a very useful maneuver, and will appear again in the study of the Klein-Gordon equation. Note that the addition of these spurious dimensions does not change the fact that the associated difference equations are numerically stable, and that the Fourier frequencies are always real. Trivial modifications of the above equations also allow one to extend the above results to spaces containing perfectly reflecting or absorbing obstacles, as well as to situations in which the speed of light, or alternately, the index of refraction, varies throughout the space. The appendix will discuss how to generalize the results obtained for the scalar wave equation to the Klein-Gordon, Maxwell, and Dirac equations.

3 Path summations

The same Monte Carlo (and related) methods that are commonly used to simulate diffusion phenomena [3–8] may be extended to the simulation of the wave equation, using the results of the previous section. Such an approach is to be distinguished from the related method of simulating lattice-wave solutions by density variations of lattice gases [4,11]. The validity of such an approach may be examined by way of the associated (linearized) Boltzmann equation [3–8,10–12]. While it is also possible to obtain a cellular automaton model of the wave equation directly and without recourse to path summations and diffusion phenomena [13], the present approach is more easily estensible to the study of Klein-Gordon and related equations relevant to physics.

We briefly recapitulate some results of the random walk as applied to the diffusion equation, in order to emphasize the similarity of that formalism to the present one. The exposition given here is presented in such a way as to anticipate and facilitate its subsequent application to the wave equation.

Consider a discrete dynamical system consisting of particles executing random walks on an orthonormal n-dimensional lattice. Henceforth, such particles will be referred to as *tokens*, to distinguish them from the particles whose wave functions will later be simulated by a gas of such tokens. At any step in the (discrete) evolution of the system, tokens are to be found at some lattice point, and they move to a randomly chosen nearest neighboring point in the subsequent time step.

The probability that a token initially at the lattice point $\mathbf{x}_0$ will at time T be found at $\mathbf{x}$ may then be written as

$$\mathrm{Prob}(\mathbf{x};T|\,\mathbf{x}_0;0) = \sum_I W(I) \ , \tag{20}$$

where I is an indexing of the set of $(2n)^T$ lattice *paths* originating at $\mathbf{x}_0$, ending at $\mathbf{x}$, and containing T steps. If one assumes that the probability $p_{j\pm}$ for taking a step along a given direction $x_{j_\pm}$ is everywhere constant, then the weighting factor $W(I)$ has the value

$$W(I) = (p_{1_+})^{r_{1+}(I)}(p_{1_-})^{r_{1-}(I)}\cdots(p_{n_+})^{r_{n+}(I)}(p_{n_-})^{r_{n-}(I)} \ , \tag{21}$$

where $r_{i_\pm}(I)$ is the number of steps along the $x_{i_\pm}$ direction that are found in the I^{th} path.

In the case where all the $p_{i_\pm}$ are equal, the right-hand side of (20) can readily be shown to converge in the continuum limit to the kernel for the n-dimensional diffusion equation [14].

Lattice ensembles

One can use (20) to give diffusion phenomena a statistical implementation. By using an *ensemble* of appropriately initialized lattices (with the initialization procedure to be discussed below), one can use the distribution of tokens on these lattices to simulate a solution of the diffusion equation $f(\mathbf{x},t)$. Like any solution of the wave equation that is to be simulated via the present formalism, $f(\mathbf{x},t)$ is assumed to be approximately constant over the length of the lattice spacing (and over any time interval the length of the fundamental time increment). It is also assumed to be bounded, normalized so that its maximum value is initially unity, and for now, positive.

Let the number of lattices in the ensemble be some very large number M. Define $n_j(\mathbf{x})$ to be the number of tokens at the point $\mathbf{x}$ at time t in the j^{th} lattice, where j ranges from 1 to M, and where the time dependence will customarily be understood.

To say that at time t the *statistical amplitude* at $\mathbf{x}$ is $f(\mathbf{x},t)$, is to say that

$$f(\mathbf{x},t) = \frac{1}{M}\sum_{j=1}^{M} n_j(\mathbf{x}) \ , \tag{22}$$

regardless of how the occupation numbers vary from lattice to lattice. Likewise, it will be said that an ensemble of lattices *statistically simulates* the function $f(\mathbf{x},t)$ if the above relation holds. (In any practical implementation, the above equals sign must be interpreted to mean 'approximately equals, to the desired degree of accuracy.')

Initialization

Next, consider how to initialize the ensemble of lattices corresponding to $f(\mathbf{x},0)$, beginning with the following definition. Performing an action A *'with probability p'* is defined as first obtaining a random number ζ, uniformly distributed between 0 and 1. If $\zeta \leq p$, then action A is performed; otherwise, it is not. The random numbers ζ obtained from multiple repetitions of such actions are assumed to be statistically independent.

Each lattice of the ensemble is to be initialized independently of the others. At point $\mathbf{x}$ of say, the j^{th} lattice, one places a token there 'with probability $f(\mathbf{x},0)$'. One then repeats this procedure for every other point of the lattice.

Every lattice in the ensemble, with j ranging from 1 to M, is initialized in this same way. This of course means that in general, there will be more than one token per lattice. (Indeed, one could in this case have chosen simply to place the tokens from all the M lattices onto one single lattice, but again, the exposition given here is made in such a way as to facilitate its application to the wave equation.)

Let us suppose that at time T, an ensemble of lattices statistically simulates $f(\mathbf{x},T)$. In each subsequent time step, let each token move to one of its $2n$ nearest neighbors, the choice being made randomly for each token. Assume there is no restriction on the number of tokens that can be found at a given point on any lattice.

By using basic probability theory, one may use (20) to show that an ensemble of lattices initialized according to the preceding procedure will continue to statistically simulate $f(\mathbf{x},t)$ at each subsequent time step. In order to simulate phenomena lasting T time steps, a number of lattices on the order of $(2n)^T M_0$ will be required in general, where M_0 is the number of lattices required to initially statistically simulate a given solution to the desired degree of accuracy.

Extensions to complex solutions

There is, of course, nothing about the diffusion equation that requires the solutions to be real. Suppose that each lattice token is endowed with an additional degree of freedom corresponding to a discrete phase factor, having one of the four possible values of 1, $+i$, -1, and $-i$; tokens in these respective phases will respectively be referred to as being positive, posimaginary, negative and negimaginary.

Assuming that at the lattice point $\mathbf{x}$ of the j^{th} lattice there are a positive tokens, b posimaginary tokens, c negative tokens and d negimaginary tokens, let the definition of the occupation number $n_j(\mathbf{x})$ be modified so that

$$n_j(\mathbf{x}) = (a-c) + i(b-d) \; ; \tag{23}$$

therefore, this 'occupation number' is now in general a complex integer. Because a, b, c, and d appear in (23) only as the differences $(a-c)$ and $(b-d)$, the statistics will not change if oppositely phased tokens (positive vs. negative, posimaginary vs. negimaginary) found at the same arc are assumed to annihilate each other, leaving behind tokens of at most two phases; this assumption will be made throughout.

One can then use ensembles of lattice tokens to statistically simulate complex solutions as well. To initialize the ensemble of lattices to correspond to the solution $f(\mathbf{x},t)$, where $f(\mathbf{x},t)$ may now be complex, one first defines the four positive functions:

$$\begin{aligned} f_{\mathrm{Re}\pm}(\mathbf{x},t) &= \pm\tfrac{1\pm\mathrm{Sgn}(f(\mathbf{x},t))}{2}\mathrm{Re}[f(\mathbf{x},t)] \\ f_{\mathrm{Im}\pm}(\mathbf{x},t) &= \pm\tfrac{1\pm\mathrm{Sgn}(f(\mathbf{x},t))}{2i}\mathrm{Im}[f(\mathbf{x},t)] \; , \end{aligned} \tag{24}$$

after which one simply initializes the ensemble according to each of the four functions simultaneously, in each case using the correspondingly phased tokens; positive tokens for $f_{\mathrm{Re}+}$, posimaginary tokens for $f_{\mathrm{Im}+}$, and so on. (It is assumed that $f(\mathbf{x},0)$ has been normalized so that none of the maxima of the above four functions exceed unity.) Therefore, at the end of this initialization every lattice will in general contain all four kinds of tokens, though at any given point on the lattice there will be tokens of at most two phases.

Applications to the wave equation

To statistically simulate a solution of the wave equation, one must first expand the point solution (which as stated previously, is specified by its values on the *points* of the lattice at two subsequent time steps) into an arc solution, by way of the Green's functions (13). It is convenient to normalize the solutions so that the (discrete) integral of the squares of the arc amplitudes is unity, i.e.,

$$\sum_{\{\mathbf{x},\sigma\}} |\psi_\sigma(\mathbf{x},t)|^2 \; = 1, \tag{25}$$

given that this quantity is conserved. (The summation indices denote that the summation is taken over all the nearest neighbor arcs of the lattice.)

Then, one initializes the lattices as in the case of the diffusion equation, except that instead of placing tokens at a *point* $\mathbf{x}$—with a probability and phase dependent on the amplitude at that point—one now places tokens in the *arcs* leading out from $\mathbf{x}$, in likewise accordance with the amplitude at those arcs. A token in the σ arc of the point $\mathbf{x}$ will then be assumed to execute a step in the σ direction in between the times $t = 0$ and $t = 1$, where the $2n$ possible values of σ again represent positive or negative directions along the axes of the lattice.

Just as in the above section dealing with traveling wave solutions, the lattice points should here again be viewed as black boxes, into which tokens enter, and out of which other tokens are generated. Explicitly, the dynamics is such that a token taking a step terminating at some lattice point will cause that lattice point to *generate* tokens in all the outgoing arcs (with probability distributions to be discussed below). In other words, the token numbers will no longer be constant, so that it is only by averaging that one recovers the conservation of amplitude and its square that is implied by the unitarity of the transition matrices.

The tokens in the outgoing arcs will in the subsequent time step travel along those arcs to the corresponding nearest neighboring lattice point, where the generating process will be repeated. (The parent token is assumed to annihilate after reaching its destination lattice point.)

Consider next the transition matrices for the n-dimensional generalization of (15). Let $|c_{\sigma,\sigma'}|$ designate the *probability* that a token coming into a lattice point along the σ' arc will produce an outgoing token in the σ arc. If $c_{\sigma,\sigma'}$ is positive, then the outgoing tokens will have the same phase factor, or sign, as the incoming tokens. If it is negative, the outgoing tokens will have the *opposite* phase of the incoming tokens.

For example, in the two-dimensional case (Figure 4), a positive token coming into a lattice point along the x_+ arc will produce 'with a probability 1/2' a positive token in the y_+ (or y_-) arc. It will also produce produce 'with a probability 1/2' a *negative* token in the x_- arc, this change of sign being mandated by the fact that the coefficient c_{x_-,x_+} is negative. The generalization to other dimensions is straightforward.

It is assumed that a parent token generates output tokens in the outgoing arc σ completely independently of the tokens it produces in any other arc σ'. Moreover, if there is more than one token entering a lattice point, the tokens emitted because of incoming token A are generated independently of the tokens emitted because of some other incoming token B. (Once the tokens are generated, it is again assumed that oppositely phased tokens found simultaneously in any *arc* annihilate each other.)

An ensemble of lattices that initially simulates the wave equation will then continue to do so in subsequent time steps, as may be shown by the same calculation as in the case of the diffusion equation. If an ensemble of size M is initialized to simulate the wave solution $\psi(\mathbf{x}, 0)$, then in order to obtain the (point) solution at any other time T, one allows the ensemble of lattices to evolve for T time steps and then obtains the quantity

$$(1/M) \sum_{j=1}^{M} n_j(\mathbf{x}, T) \tag{26}$$

where $n_j(\mathbf{x}, T)$ now refers to the sum of the occupation numbers for the arcs leading into the point $\mathbf{x}$ (at time T in the j^{th} lattice). Again, simulating phenomena lasting T time steps to within an initially prescribed accuracy will require a number of lattices on the order of $(2n)^T M_0$, where M_0 is as before. Note that even though the wave equations are time-reversal invariant, the dynamics used in their simulation are asymmetrical with respect to time-reversal, and it is only by averaging that the symmetry is recovered.

The relationship between amplitudes and paths on the lattice that exists in the case of the diffusion equation may be retained in the present case. A token coming into $\mathbf{x}$ along the σ arc that produces an outgoing token in the σ' arc still specifies a path increment; any token generated in another outgoing arc likewise represents the increment of yet another path. Given that in the general n-dimensional case the coefficients of the transition matrices are $1/n$ and $(1-n)/n$, one may heuristically say that the (generalized) probability of a token making a path increment along an incoming arc σ to an outgoing arc σ' is $1/n$, unless the two arcs are oppositely oriented (implying that the associated path increment is a 'reverse step', or *reversal*), in which case it is $(1-n)/n$.

The relation (20) has an analogue in the case of the wave equation, in that the hodotic solutions (and, given the completeness thereof, any solution of the wave equation) can similarly be expressed in terms of summations over paths. In fact, in accordance with the considerations of the previous paragraph, the n-dimensional hodotic solution has the likewise expansion

$$h_\sigma(\mathbf{x}-\mathbf{x}_0, T) = \sum_{I_\sigma} W(I_\sigma) \tag{27}$$

where the $2n$ possible values of σ again represent positive or negative directions along the lattice, and where I_σ is an indexing of lattice paths of length T whose initial step is along the σ direction. In the present case,

$$W(I_\sigma) = (1/n)^T (1-n)^{R(I_\sigma)} \tag{28}$$

where $R(I_\sigma)$ is the number of reversals in the path I_σ; for the one-dimensional case $n=1$, zero to the zeroth power is defined to be one.

Thus, there exists a statistical representation of the flows and traveling waves of the wave equation. The path summation formalism is retained, except that the paths are converted into chains of causality with the result that the Huygens picture of wave propagation, whereby any portion of a wave itself emits a wavelet, is likewise given a statistical rendering.

4 Quantum Mechanics

The lattice systems considered here, with their ability to reproduce wavelike phenomena by way of discrete particle motions, provide a novel way of simulating quantum mechanical systems, as the present section will show. As a first step, a lattice quantity will be introduced whose expectation is equal to the absolute square of the wave function under consideration, and which is additive in the number of ensemble members.

Initially, the wave functions will for convenience be assumed to satisfy the lattice wave equation, so that they can be interpreted as the wave function of some massless and spinless particle; moreover, the discussion will at first be restricted to one-particle systems.

Bilattices

The phased tokens used in the previous section to statistically model a solution to the wave equation ψ can obviously be used to obtain the square of the solution as well. However, by using *two* independent sets of tokens—one to represent ψ and one to represent ψ^*—it is possible to obtain $|\psi|^2$ in a more direct fashion, with the duality between the two sets of tokens being in direct correspondence to that which exists between the bras and kets of Dirac's formalism.

Consider again a lattice that has been initialized to simulate some wave function $\psi(\mathbf{x}, 0)$. Next, use a differently labeled, but otherwise identical set of tokens to initialize the identical wave function on that same lattice—note that because the initialization procedure is nondeterministic, the actual configurations of the two sets of tokens on the lattice will in general be different. The first set of tokens will be referred to as *bra* tokens, the other as *ket* tokens, and the lattice containing these two types of tokens will be called a *bilattice.*

Let $\hat{n}(\mathbf{x})$ refer to the (complex) occupation number, as defined above, obtained by counting the ket tokens in all the arcs leading into the point $\mathbf{x}$; similarly, let $\check{n}(\mathbf{x})$ refer to the occupation number obtained by counting the bra tokens on the same arcs. The motion of the bra tokens is totally independent of that of the ket tokens.

Event counting

Although there is no direct correspondence between lattice tokens and the particles of matter whose wave functions are being simulated, there does exist a quantity, expressed in terms of the location of tokens at a lattice space-time point, whose expectation is equal to the probability of finding a matter particle at the corresponding space-time point in the physical space being simulated.

By way of analogy, the simultaneous presence of any bra token and any ket token at some lattice point $\mathbf{x}$ will formally be referred to as a one-particle *event,* for reasons which will become clear below, and will be denoted as $e(\mathbf{x})$ (the time dependence is suppressed for clarity). Taking the summation of such events over an arbitrary set of lattice points will likewise be referred to as a one-particle event count, even though the increments to this count need not be positive, and may have an absolute value greater than one. Every event has a phase factor associated with it, which is defined to be the phase factor of the ket token times the complex conjugate of the phase factor of the bra token, and the event count is incremented algebraically according to the phase of the respective events. Thus, if $\check{A}$ denotes a bra token and $\hat{B}$ denotes a ket token, one may express the phase factor of the event for which a bra token $\check{A}$ and a ket token $\hat{B}$ arrive at $\mathbf{x}$ as

$$\mathrm{Ph}[e(\mathbf{x})] = \mathrm{Ph}[\check{A}]^*\mathrm{Ph}[\hat{B}] \tag{29}$$

Typically, the distinction between $e(\mathbf{x})$ and the weighting factor $\mathrm{Ph}[e(\mathbf{x})]$ will be ignored, and the former expression will be used to designate the latter. If j bra tokens and k ket tokens arrive at some space-time point $(\mathbf{x}, t)$, they collectively represent jk distinct one-particle events, so that one may write, by way of definition,

$$\sum e(\mathbf{x}) \equiv \check{n}(\mathbf{x})\hat{n}(\mathbf{x}) \ , \tag{30}$$

where the summation is over all the one-particle events at $\mathbf{x}$ (and t). Now the initialization of bra tokens is statistically independent of the initialization of the ket tokens. Therefore, on any bilattice which has been initialized to simulate the wave function $\psi(\mathbf{x}, 0)$, it is the case that

$$\check{n}^*(\mathbf{x})\hat{n}(\mathbf{x}) \doteq \psi^*(\mathbf{x}, t)\psi(\mathbf{x}, t) \ ,$$

where the count has been taken at t.

Note that bra and ket tokens will increment the event count taken at a given point even if they do not come into that point on the same arcs, so long as they are both on the same bilattice. Also note that since the expected value of the event count is always real, one may as well ignore any events having an imaginary phase, so that with such an understanding, the results of the event count can be considered a real quantity.

Let $\check{n}_\sigma(\mathbf{x})$ denote the occupation number of ket tokens found in the σ arc in a bilattice leading into the point $\mathbf{x}$, and $\hat{n}^*_\sigma(\mathbf{x})$ likewise denotes the complex conjugate of the bra occupation number of that same arc. It then follows that the expected value of $\check{n}^*_\sigma(\mathbf{x})\hat{n}_\sigma(\mathbf{x})$, when integrated over all the arcs in the bilattice, is conserved, due to the unitarity of the transition matrices. This quantity is to be distinguished from the expected value of the summand in (30). The latter is obtained by considering tokens arriving at *points,* and its discrete integral over time varies, while the former (constant) quantity is obtained by considering the location of tokens on the individual arcs.

Diffraction

One can of course initialize bilattices to simulate to a wave packet of some well defined momentum. Consider the following double-slit diffraction experiment such as is found in standard introductory quantum mechanics texts. Imagine placing in the bilattice a screen, facing orthogonally to the momentum of each wave packet. Assume the screen contains two parallel slits having widths on the order of the mean wavelength of the wave packet, with the slits being separated by a distance smaller than the coherence length of the packet. Any convenient boundary conditions may be imposed on the slits—for example, those corresponding to perfect absorption.

Consider making an event count at some observation point $\mathbf{x}$ on the opposite side of the screen at some distance that is large in comparison to the widths of the slits, and which a lattice particle can only reach by propagating through those slits. Quantum mechanics tells us that the probability of a particle being found at a point $\mathbf{x}$ of the observation screen at some time t is

$$|\psi_{\rm I}(\mathbf{x}) + \psi_{\rm II}(\mathbf{x})|^2 \ ,$$

where $\psi_{\rm I}$ is the amplitude whose square would determine the probability of a particle arriving at $\mathbf{x}$ if only slit I were present, and so on. The time dependence of the wave functions will be understood.

Suppose that a large number of identical lattices are initialized to simulate some function $\psi(\mathbf{x})$. Consider the event count taken on all the points corresponding to some point $\mathbf{x}$ for which complete cancellation occurs, i.e., where the amplitude associated with one slit totally cancels the amplitude associated with the other.

Quantum mechanics predicts that in such a region, no particles will arrive. In the present model, however, there are in general many increments in what is called the event count; the result of the count would then be zero (or, more precisely, vanishingly small in comparison to the counts taken in a region of constructive interference) only because positive events are equally likely as negative events.

The absolute square of wave function can be written as

$$\psi_{\rm I}^*\psi_{\rm I} + \psi_{\rm II}^*\psi_{\rm II} + \psi_{\rm I}^*\psi_{\rm II} + \psi_{\rm II}^*\psi_{\rm I} \ , \tag{31}$$

where the dependence on time and space has been suppressed. Note that the events, i.e., the simultaneous arrivals of a single bra and a single ket token, can be conceptually placed into four classes, corresponding to each of the above four terms. Respectively, a token from the first class represents events in which both the bra and ket token came through slit I; the second, an event for which both came through slit II; the third, an event for which the bra token came through slit I and the ket came through slit II; and the fourth, an event for which the bra token came through slit II and the ket came through slit I. (Saying that the bra token of a certain event came through a certain slit means that the chain of interactions that led to the token's arrival at the observation point define a space-time path that leads through that particular slit. Note again that several tokens can be found at any bilattice arc, each one being associated in general with a distinct path.)

Depending on how the slits and the observation points are situated, one may recast all the features of the particle diffraction *gedankenexperiment* in terms of event counts obtained on the ensemble of bilattices. For example, shutting one of the slits, or slightly perturbing the tokens exiting from either slit can block the arrival of certain events, or destroy the coherence evidenced by the third and fourth terms in the above expression.

The interference patterns shown in Figure 6 were obtained by simulating a double-slit diffraction experiment by way of the discrete, statistically implemented two-dimensional lattice wave equation (2). The dimensions of the bilattice are 33×75

Figure 6: Interference patterns obtained in a two-dimensional double-slit diffraction experiment. Dashed lines show the event count distributions subsequent to the emission of 50, 100, 150, and 200 million "photons", as a function of position along the observation screen. The solid line indicates the exact solution of the interference pattern corresponding to the lattermost event count distribution.

units, with the observation screen, i.e., the line of points along which the event counts are taken, being oriented along one of the long edges of the bilattice. On the edge opposite to the screen are the one-unit-wide slits (i.e., the emission points) which are situated ± 10 units from the center. The wave packets, which were each given the standard normalization indicated in (25), were such that their contributions to the bra and ket occupation numbers at the slits were integers whose expected values varied sinusoidally, with a period of 10 time steps. The total time duration of each packet was 50 time steps. The nondeterministic evolution of the token occupation numbers at any arc was obtained simply by iterating the arc amplitudes at any time step according to (16) and then incrementing any resultant half-integer occupation number by a random choice of $\pm 1/2$, thereby ensuring that the token occupation numbers were always integers. The remaining boundary conditions were such as to correspond to total absorption of the tokens at any of the lattice's edges.

The four dashed lines show the event count distribution after the emission of 50, 100, 150, and 200 million "photons", respectively, with the abscissa ranging along the one-dimensional observation screen, from 0 to 75. Each tick on the ordinate axis corresponds to a million events. The solid-line graph—which has been normalized so as to have the same integral as the graph corresponding to 200 million photons—shows the exact solution of the interference pattern, obtained by integrating (2) with real (as opposed to integer) arc amplitudes and emissions.

Note for future reference that it is possible to make the amplitudes in even this discrete formulation quasi-continuous by assigning each bra and ket token a fractional weight of γ, where γ is a small positive number. Then the arrival of a bra and ket token would only increment the event count by a factor of $\pm\gamma^2$, where the sign depends on the phase of the tokens. As the value of γ is made to approach infinitesimal values, the amplitudes take on quasi-continuous values; nevertheless, as will be shown, the way in which the mass and potential terms are incorporated into the formalism implies that the amplitudes are ultimately best considered as discrete entities rather than continuously varying quantities.

4.1 Multi-particle phenomena

Consider next the simulation of multi-particle phenomena.

Assume for now that all the particles (as opposed to tokens) being simulated are indistinguishable, and are fermions; the case of simulating bosons will follow straightforwardly. Although the systems under consideration in this paper are relativistically covariant (in the continuum limit), the number of particles will still be a good quantum number in the absence of interactions, so that it is permissible to speak of multi-particle states containing a fixed number of particles.

In considering multi-particle wave functions, it is necessary that the tokens corresponding to different one-particle states be distinguishable from the tokens corresponding to another. There are a number of ways to impose this distinguishability, some more physically plausible than others, but for the sake of the presentation, it is simplest to assume that the tokens corresponding to different states are distinguished by a label or a tag of some sort, which shall be referred to as a *particle index*. In fact, this particle index can be viewed as a sort of rudimentary momentum, given that it represents an extra degree of freedom that a particle may assume. However, it does not influence the dynamics of a token. The range of the particle indices, i.e., the number of distinct values they can assume, needs to be only as large as the number of one-particle states that are being simulated.

Initialization

Consider the Slater determinant of the N one-particle states $\phi_1(\mathbf{x}), \ldots, \phi_N(\mathbf{x})$ by way of which the N-particle wave function is expressed in terms of its one-particle state constituents $\phi_i(\mathbf{x})$, with i ranging from 1 through N, as

$$\psi(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \frac{1}{\sqrt{N!}} \sum_P \phi_{P[1]}(\mathbf{x}_1) \cdots \phi_{P[N]}(\mathbf{x}_N) \sigma(P) \tag{32}$$

where P is some permutation of the numbers 1 through N, the sum is over all such permutations, and $\sigma(P)$ is 1 or -1 according to whether the permutation is even or odd. (Again, the time dependence of the wave functions has been suppressed.)

To initialize a lattice to simulate such a wave function, one can proceed as if the wave functions possessed no particle exchange symmetry. That is, the wave function is initialized according to *one* of the $N!$ terms in (32); the symmetrization will ultimately appear as a result of the way in which the count is taken.

For convenience, consider initializing a lattice according to the wave function

$$\phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2) \cdots \phi_N(\mathbf{x}_N) . \tag{33}$$

In doing so, each one-particle wave function is to be initialized independently; moreover, all the tokens used in initializing $\phi_j(\mathbf{x}_j)$ will have identical particle indices, that are distinct from those used in initializing some other state $\phi_k(\mathbf{x}_k)$. Therefore, presuming that the single-particle wave functions will in general overlap, each arc will be allowed to contain tokens with differing particle indices. Later on, it will be shown that for fermion states, the number of distinct particle indices at any single arc can be reduced to one. For now, it will be assumed that only oppositely phased tokens with the same particle index are allowed to annihilate.

An $\tilde{N}$-particle event (where $\tilde{N}$ need not equal N) is then defined as the simultaneous arrival of $\tilde{N}$ distinctly labeled tokens at the observation points $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_{\tilde{N}}$, along with the arrival of $\tilde{N}$ ket tokens, subject to the restriction that the particle index of the ket token found at $\mathbf{x}_i$ is the same as that of the bra token at $\mathbf{x}_{P(i)}$, where P is some (single) permutation of the numbers 1 through $\tilde{N}$. Such an event will be denoted as $e(\mathbf{x}_{P[1]}, \mathbf{x}_{P[2]}, \ldots, \mathbf{x}_{P[\tilde{N}]})$, or $e(P[1], \ldots, P[\tilde{N}])$ for short. The phase of the event is defined to be the product of all the ket tokens' phases times the complex conjugate of the product of the bra tokens' phases, times an additional factor of -1 if P is an odd permutation.

By arguments similar to those considered in one-particle systems, the expected $\tilde{N}$-particle event counts are such that

$$e(P[1],\ldots,P[\tilde{N}]) \doteq \sum_{i_1=1}^{N} \cdots \sum_{i_{\tilde{N}}=1}^{N} \phi^*_{i_1}(\mathbf{x}_1)\phi_{i_1}(\mathbf{x}_{P[1]}) \cdots \phi^*_{i_{\tilde{N}}}(\mathbf{x}_{i_{\tilde{N}}})\phi_{i_{\tilde{N}}}(\mathbf{x}_{P[\tilde{N}]})\sigma(P)\Bigg|_{i_j \neq i_k} , \tag{34}$$

where it is assumed that the bilattice in question was initialized to statistically simulate the N-particle wave function $\psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$, as expressed in (33), and where the summation is restricted to terms for which all the i_j are distinct. This quantity is equal to the probability

$$\mathrm{Prob}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{\tilde{N}}) , \tag{35}$$

of finding $\tilde{N}$ particles at the respective observation points times a factor of $\tilde{N}!$, this factor resulting from the fact that the present formalism yields only the sum of probabilities of the form (35) that are identical up to a permutation of its arguments. Alternatively, one could have chosen to distinguish each possible assignment of particle indices to the tokens during assignment.

The validity of (34) is most easily proven by showing that events for which an identically tagged bra and ket particle are located at the respective observation points $\mathbf{x}_i$ and $\mathbf{x}_j$ have an expected value proportional to $\sum_{k=1}^{N} \phi^*_k(\mathbf{x}_i)\phi_k(\mathbf{x}_j)$. It may likewise be shown that the above result holds for arbitrary N and $\tilde{N}$, provided that the summation is taken over all ordered selections (without repetition) of $\tilde{N}$ elements from the set $\{1, \ldots, N\}$; note that from the definition of multi-particle events, it follows that the expected value of the event count is identically zero in the case where $\tilde{N} > N$.

As an example, let the diagram

$$\begin{array}{cccc} G & B & R & O \\ \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 \\ R^* & O^* & B^* & G^* \end{array} \tag{36}$$

stand for the four-particle event $e(4312)$ in which ket tokens whose particle indices shall be labeled 'green' (G), 'blue' (B), 'red' (R) and 'orange' (O) arrive at the four observation points $\mathbf{x}_1$ through $\mathbf{x}_4$ simultaneously with bra tokens labeled 'red', 'orange', 'blue' and 'green', respectively. The phase of this event is then the product of the ket phases times the complex conjugate of the bra phases, times another factor of -1, because $ROBG$ is an odd permutation of the labels $GBRO$. Note that while the interchange of any two bra (or else ket) tokens negates the phase of an event, exchanging any two particle indices leaves the phase unaltered, since this is equivalent to an interchange of both bra and ket tokens.

An N-particle event involving distinguishable particles, say particles of type A and B, will be considered valid if bra and ket tokens of type A constitute a valid

N_A-particle event and bra and ket tokens of type B constitute a valid N_B-particle event, where $N = N_A + N_B$. The phase of the event is the product of the phases of the constituent events. Events involving bosons are treated in the same fashion, except that the phase of an event is obtained directly from the bra and ket tokens, without any extra factor of $\sigma(P)$.

Note that it is possible to initialize distributions of N particles which give a predetermined N-particle event count while at the same time ensuring that all $\tilde{N}$-particle event counts for which $\tilde{N} \leq N$ have an expected value of zero. For instance, in a 2-particle event initialized with tokens having the particle indices red (R) and blue (B) respectively, consider the operation of multiplying all the R ket tokens by a phase of i^r and all the B tokens by a phase of i^b where r and b are integers such that their sum is a multiple of 4. If $r = 1$ for example, this means transforming all the positive R tokens into posimaginary ones, all the posimaginary ones into negative ones, and so on. A similar transformation is applied to all the constituent bra tokens as well, with some other randomly chosen integers r and b (similarly subjected to the constraint that their sum a multiple of 4). In any ensemble of identically prepared two-particle systems, let this randomization be repeated throughout.

Clearly, any two-particle event count is invariant under such an operation, since the phase of every two-particle event involves the *product* of the constituent R and B tokens, and this product remains unchanged, for both bra and ket tokens. However, because there is now an essentially random phase relationship between the R bra and the R ket tokens, etc., any one-particle event count will be zero. By modifying the definitions of r and b for either the bra or ket states, one can construct two-particle events whose expected values are negative.

Such anomalous distributions of tokens, and their generalizations, are not simply a mathematical artifice; they are in fact necessary in order to properly initialize systems of particles with internal structure (spin, isospin, etc.) However, since measuring such properties involves subjecting the particle states to the corresponding mediating fields—and thereby elaborating the requisite interactions—the utilization of such distributions will be left as a future development.

Coalescing token tags

As mentioned previously, the Pauli exclusion principle allows for a considerable simplification of the token dynamics, in the case of fermionic wave functions.

Consider as an example a two-particle state, for which the particle indices will again be denoted as red (R) and blue (B). Because the wave function at any point corresponding to either of the single-particle states can be written as a sum over token paths, the product of the two wave functions at any two points may likewise be written as a sum over pairs of token paths. Consider all such pairs of paths for which two distinctly tagged ket tokens are at some time within the same arc. Thus, the R and B tokens may be said to travel into the same arc, whereupon exiting, the R token eventually gets to, say, the observation point $\mathbf{x}_1$ at the same instant as the

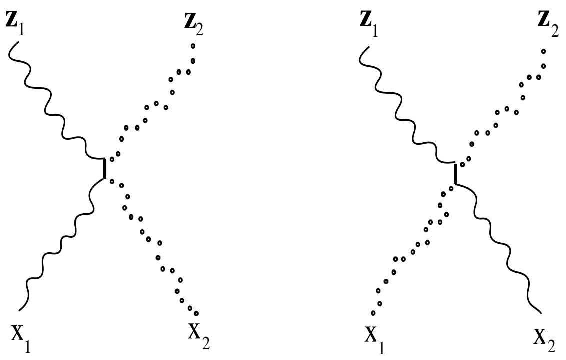

Figure 7: Products of hodotic wave functions can be related to a sum over pairs of paths. Any pair of paths which coincide at some arc can be matched with a pair whose steps subsequent to entering the arc are interchanged. The net contribution of such pairs is zero.

B token gets to $\mathbf{x}_2$ (cf. Figure 7).

Note that because the two tokens were initially on the same arc, the probability of one of them subsequently arriving at a given point with some phase is the same for both tokens (up to a factor representing their phase differences at the time when they were both in the same arc). In other words, for any pair of paths in which the R token arrives at $\mathbf{x}_1$ and the B token arrives at $\mathbf{x}_2$, there is another equally probable pair of paths, call it the switched pair, for which the R token arrives at $\mathbf{x}_2$ while the B token arrives at $\mathbf{x}_1$, i.e., for which the tokens exchange their paths subsequent to entering the common arc.

The first pair of paths appears in the path summation expression of $\phi(\mathbf{x}_1)\chi(\mathbf{x}_2)$, while the switched path appears in the expression of $\chi(\mathbf{x}_1)\phi(\mathbf{x}_2)$.

Now any associated event for which the two ket paths have a path step in common is just as likely as an event for which the bra paths are identical, but for which the ket paths are the switched paths. But since $\sigma(P)$ differs for the two events, they will have opposite signs, so that the expected contributions of such events cancel. Therefore, in the context of this model, the effect of Pauli exclusion principle is a cancellation of all events involving the simultaneous presence of distinctly indexed tokens at any arc.

As a result, it is possible to greatly simplify the token dynamics. One cannot simply have such coincident ket tokens of opposite phase annihilate; that would distort the event counts involving only events in which there is, say, only a single *bra* token with a particle index of R. A similar argument precludes the simple annihilation of ket tokens.

Rather, it is the case that whenever two tokens of differing particle indices are found in the same arc, they may be replaced by two tokens whose particle indices are identical. That is, one can have either of the tokens take on the particle index of the other. If there are only two tokens present, then the surviving particle index has an equal probability of being either of the two initial values. In the above example, rather than having an R token and a B token exiting an arc with probability one, there can instead be two R tokens or else two B tokens, each with a probability of $1/2$. Therefore, the statistical weight attributable to each of the path steps in question remains unaltered.

In order for this dynamical refinement to produce the proper sign, a token that assumes another token's particle index must also assume the phase of that other token. As an example, suppose a positive R token and a negative B token arrive at a given arc. In that case, the exiting tokens will both be positive, and have the particle index R, or else will both be negative, with a particle index of B; the probability of either outcome is $1/2$.

This procedure may be generalized to any distribution of tokens. Suppose that j bra tokens of particle index R, and phase ϕ_R, and k bra tokens of particle index B and phase ϕ_B are found in a single arc. (It is assumed that all oppositely phased token with identical particle indices have already been annihilated.) Then with probability $j/(j+k)$ all the tokens will be given the particle index R and the phase ϕ_R, or with probability $k/(j+k)$ they are all given the particle index B and the phase ϕ_B. In the general multi-particle case, when tokens of more than two particle indices may be found in the same arc, it is easy to show that the tokens corresponding to any two particle indices may be coalesced into a single particle index first, whereupon they may be coalesced with the tokens corresponding to some other particle index, and so on. The coalescing may even be extended to sets of identically indexed tokens located within an arc that are in two phases. For example if there are j positive R tokens and k posimaginary R tokens, the tokens may be coalesced into being either entirely positive R or entirely posimaginary R tokens, with the probability of either outcome being respectively equal to $j(j+k)$ or $k/(j+k)$.

The ability to coalesce the identities of coincident tokens greatly simplifies the model. Instead of there being a large number of particle indices at each arc in the lattice, there will be only two (one for the all the bra tokens and one for the ket tokens). Thus, at any arc of the bilattice, there needs to be a record only of the number of the bra tokens there, and if that number is nonzero, the phase and the particle index of all the tokens; the same holds for the ket tokens. Of course, information is lost by coalescing tags and phases, so that a larger ensemble of bilattices will be necessary to obtain a predetermined accuracy in any results.

It is also possible, in the case of antisymmetrized wave functions, to perform multi-particle event counts without having to consider particle indices—in other words, in a formalism for which all tokens are truly indistinguishable, apart from their phases and classifications into bra and ket particles. In such a case, however, the event count necessarily involves keeping track of the tokens arriving at the observation points in a sub-ensemble of bilattices, of order at least N. The proper particle exchange symmetry then comes about by having unwanted events cancel, on the average, due to the incoherence imposed on their phases.

5 Macroscopic systems

This section will discuss the features of the present formalism which are effectively unobservable in the macroscopic realm. Presuming one wishes to simulate some macroscopic system, and that the wave function of every participant particle of the system is prescribed, one initializes a bilattice by placing bra and ket particles at a given arc—just as before—in accordance with the probability amplitude at any arc of the space in question. As previously mentioned, in systems containing more than one kind of particle, say electrons and quarks, there need to be different types of bra and ket tokens for each of the different kinds of particles.

Subsequent to the initialization, and presuming that the interactions among all the particles are properly approximated by the dynamics of the lattice tokens, the simulation proceeds by simply allowing the system to evolve for the desired length of time and then taking event counts in order to observe the resultant distribution of particles. For present purposes, it is assumed that event counts will be taken along a sequence of nearest neighboring lattice points which converge, in the continuum limit, to a space-time path that may be traversed by a classical particle.

Note also that it is adequate for the present purposes to consider only event counts of one type of particle, say the electrons, since in principle, one can infer the behavior of all the other particles from the changes they affect in the distributions of electrons, or some other preferred particle [15]. Now in order to obtain from this formalism a model of macroscopic phenomena, one must mandate that all questions that can be asked of the system (that is, all the measurements made on the system) must be answered by way of event counts; such a restriction is fundamental to what follows. (In the case of multi-particle event counts, for which the N associated observation points trace N distinct space-time paths, it is assumed that every event count increment occurs at a single preferred point, at which information regarding the location of the event's participant bra and ket tokens has arrived. The multi-particle event count is then assumed to be incremented along a pathlike sequence of preferred points which also may be connected by a classical space-time path.)

Moreover, any event count is assumed to be defined only to within an accuracy of $\pm M$, where M is some number large in comparison to unity. As discussed previously, the fluctuations of bra and ket occupation numbers at any point about their expected values depend on the specific dynamics chosen for the tokens, as well as on the magnitude of the weight given to each bra and ket token. For small weights (and thereby small deviations), M may also be made smaller, and with such an understanding, its exact value may be left unspecified.

The introduction of the parameter M is due to the fact that measurements in this formalism are understood to always be made by macroscopic, "classical" beings. Such beings are unable to observe microscopic phenomena directly, and must first make the microscopic states interact with larger ones, the latter having the property that a change in size of $\pm M$ particles is imperceptible. Even so, it is still possible to speak of the outcome of some single quantum event, provided that one is willing to consider simulating the larger, classical measuring device which allows that single quantum event to be perceived. This is, after all, what happens in the real world as well. For example, consider the case where one observes the track an electron leaves in a bubble chamber. What is ultimately observed in such a situation is not the single electron, but the distribution of matter in the macroscopic system that is the bubble chamber itself. In the present formalism, one can simulate the wave function of the single electron in question, but information regarding its evolution is then obtainable only by referencing (i.e., making measurements on) the macroscopic system that is the bubble chamber.

(In fact, one ultimately has to consider that what is typically being measured in the above example is not the distributions of electrons in the cloud chamber, but rather the distributions of electrons affected by photons of light coming from the bubble chamber's surroundings and impinging on the observer's eyes; but in any case, quantum phenomena are perceivable only through their effects on macroscopic systems.)

As another example, supposing that one wishes to simulate a pebble as a macroscopic collection of interacting electrons, neutrons, protons, etc. One could infer the position of this pebble by taking event counts in some localized region of space. Also, one could experimentally verify that the pebble is a localized distribution of matter by performing a two-particle event count at observation points whose separation is greater than the size of the pebble and noting that such event counts are consistently zero (plus or minus M). Other multi-particle event counts could be used to determine the moments of the mass distribution, and ultimately the shape of the pebble. Still other event counts could determine the velocity of the pebble relative to, say, some other distribution of matter being simulated, and so on. Therefore, the requirement that all information about a system must be obtained by way of macroscopic event counts is not an intractable one. Indeed, some procedure akin to the repeated and continual taking of event counts may well be the method by which human observers obtain information about their surroundings, even though such matters, for all their relevance [16–17], lie well beyond the scope of this presentation.

Incompatibility of outcomes

As previously discussed, the numbers obtained through any macroscopic event count will be compatible with the expected numbers of particles predicted by quantum mechanics, and therefore, with the classical limit thereof. However, the outcome of an event count can vary widely according to the choice of *integration path,* that is, the space-time path along which the event count is incremented. The implications of this path dependence are especially relevant in the present formalism, where a nonlocalized distribution of tokens is associated with every particle's wave function. For example, in the above case of the bubble chamber, an integration path taken in some space-time region A may correspond to an electron just having arrived at one region of the chamber; another integration path in some distant region B of the chamber may be compatible with the electron having arrived there (and implicitly, not in region A). Such anomalies, affecting even event counts taken on "classical" systems, will be encountered whenever microscopic systems are amplified so as to influence macroscopic ones. And as mentioned previously, such an amplification is precisely what is necessary in order for quantum mechanical phenomena to be observable in the macroscopic realm.

It should be noted that all of the different physical outcomes obtained by considering several different space-time paths are equally valid, and the corresponding "many worlds" are all simulated at once [18]. However, to each and every single observer there corresponds only a single integration path, and therefore only a single outcome.

Also note that a two-particle event count, taken along pairs of points that are in region A and B respectively, would be insufficient in settling the discrepancy between the two single-particle counts in those regions, because it too would merely produce an event count that is compatible with quantum mechanics, and not necessarily with either of the single-particle event counts it attempts to correlate.

Moreover, any appeal to a memory device, be it a neuron or a lab notebook, will likewise be able to resolve the matter, since memory devices are—like all other physical systems—simply distributions of particles. One could compare the present state of a system with the memory devices used to store information about its past states, but doing so would involve making correlation experiments and multi-particle event counts with the particles comprising the devices. Again, the outcomes of these event counts would be numbers that are compatible with quantum mechanics and therefore its classical limit, but they would not necessarily be compatible with other event counts made on systems which the memory devices were intended to remember.

As a result, the nonintuitive features of this formalism—nonpositive event count increments, multiplicity of tokens per particle state, etc.—are rendered operationally unobservable to any macroscopic observer whose sole source of information comes as a result of event counts. Moreover, the conceptual difficulty of becoming accustomed to such notions is compensated by the fact that the resultant model of quantum mechanics is truly three-dimensional and local, and is far simpler overall than more mathematically sophisticated formalisms.

6 Discussion

In conclusion, it is the case that a gas-like, three-dimensional distribution of discrete lattice particles, governed by a local dynamics, is sufficient to simulate multi-particle quantum mechanics. The present approach exhibits features of several other formalisms designed to relate quantum mechanics to systems that are more classical, in the sense of being able to be modeled by local and three-dimensional dynamics [19–20].

The motivation behind one such formalism is the similarity of the Schrödinger equation to the diffusion equation [21–23], despite the consequent obstacle of making the resultant systems relativistically covariant. The present approach, in contrast, begins with systems of equations that are already relativistically covariant (in the continuum limit). This means, however, that the Schrödinger equation is obtainable only as the low-velocity limit of the associated Dirac and Klein-Gordon equations.

Other approaches, which have the advantage of making the evolution of quantum mechanical systems formally equivalent to classical phase-space systems, rely on the notion of generalized probabilities [24–25]. Dirac was able to arrive at a phase-space formulation of quantum mechanics in which probability amplitudes (as opposed to simply their squares) are formally equivalent to probabilities, in the sense that small amplitudes are associated with events that occur only rarely [26]. A better known example of this approach is the Wigner function formalism, in all its manifestations and variations [26–34], though it too cannot readily be extended to relativistic phenomena. Still other approaches involve the Euclideanization of time in computations involving propagators, a procedure not necessary in the present formalism.

The interpretations formulated by Bohm, Vigier, and others [21, 36–38] and the hydrodynamical models of wave functions that preceded them, are also models of quantum phenomena derivable in terms of the flow of conserved quantities. However, such formalisms are able to accommodate the seemingly nonlocal phenomena associated with the EPR paradox [38–40] only by hypothesizing dynamical evolution taking place in a phase-spaces of dimensionality proportional to the number of particles in the system, whereas the present formalism can be cast in a three-dimensional configuration space.

Some of the mathematical concepts underlying the approach taken here have also been studied before. In particular, its similarity to a related model introduced by Feynman has already been noted. Also, S. Gudder has derived the constraints an abstract lattice system or graph must satisfy if it is to simulate quantum mechanics (and its associated wave equations) [2,41]. Still others have argued that lattices, discrete networks, and other continuum-violating paradigms have an importance beyond their utility as computational or analytical aids, given that classical notions of continuity are not expected to be valid at sufficiently microscopic scales [42–50]. Finally, as noted previously, the discussion of the applicability of the formalism to the modeling of macroscopic system implements some of the notions found in Everett's relative state formulation of quantum mechanics [18], as well as Von Neumann's and Wigner's theories of measurement [16–17].

7 Appendix

7.1 The Klein-Gordon equation

The previously obtained results for the wave equation can be extended to the computation of the Klein-Gordon equation. On an m-dimensional Euclidean space, The Klein-Gordon equation takes the form

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_m^2}\right)\psi + \eta^2\psi \ , \tag{37}$$

where η is a real parameter (and where partial derivatives have their usual continuum definitions.) The Fourier frequencies of this equation obey a dispersion relation of the form

$$\omega^2/c^2 = k_1^2 + k_2^2 + \cdots + k_m^2 + \eta^2 \ .$$

Specification of dimensions

Recall the discussion of the wave equation of a toroidal space of n dimensions, all except m of which are "thin", i.e., are of length one. Suppose now that these thin dimensions were of length two instead of one. (If the length of all the dimensions were even, then it would again make sense to speak of even and odd sublattices.) The lattice would still be effectively m-dimensional, but there would also be extra degrees of freedom to consider.

First suppose that $n - m = 1$. A thin dimension of length two implies that one more set of spatial frequencies, κ_{x_n}, would be required to specify the system, corresponding to motions along the thin dimension. However, since the index x_n only takes the values '0' or '1' (or by an appropriate change of variables '$\pm 1/2$'), it could be viewed as merely a specifier between two different "components" of the solutions.

The dispersion relation of this system would then be of the form

$$\sin^2\frac{\omega}{2} = \frac{1}{n}\left(\sin^2\frac{\kappa_{x_1}}{2} + \cdots + \sin^2\frac{\kappa_{x_m}}{2} + \sin^2\left(\frac{\pm\pi}{2}\right)\right) \tag{38}$$

(the last term of course being equal to one). For extremely small lattice spacings, this expression would become, in analogy with the discrete wave equation dispersion relation described above,

$$\omega^2/c^2 = k_1^2 + \cdots + k_m^2 + \left(\frac{\alpha}{d}\right)^2 \ , \tag{39}$$

where α is a constant on the order of one, and d is the length of the lattice spacing. This is precisely of the form of the Klein-Gordon dispersion relation, with d/α acting as the Compton wavelength of the associated particle.

This approach therefore is loosely reminiscent of Kaluza-Klein models, insofar that mass is the manifestation of the topology of the space corresponding to motion along a "matter" dimension. Of course, this system is only useful for situations involving spatial frequencies, i.e., momenta, much smaller than d^{-1}, so that only nonrelativistic phenomena can be accurately simulated.

Suppose, however, that m is the typical 2 or 3, corresponding to two and three dimensional cases, and that n is large, even astronomically so. Such a system has an evolution that is correspondingly harder to compute (because the speed of light is then very "slow", i.e., a relatively large amount of computation is necessary to execute a given time interval of the simulation). Even so, the above considerations for the case $n = m+1$ still hold, except that the mass term in the dispersion relation, α, would be multiplied by a factor of n, thereby increasing the spectral range in which the approximation to the continuous case is good. Also, the approximately n momenta corresponding to motion along the thin dimensions can likewise be further manipulated, depending on the internal substructures and symmetries of the particles one wishes to simulate.

Such a model is also reminiscent to the one-dimensional Klein-Gordon equation considered by Feynman [52–54] (which he refers to as a Dirac equation). Although the model described here has the advantage of being applicable to systems in any dimension, as opposed to simply the one-dimensional case [53–54], both versions are limited by the fact that the lattice spacing may no longer be taken to zero, but has a fixed length proportional to the Compton wavelength of the particle under consideration. Even if one could disregard the problems of accommodating such a model with the continuity of space that has been observed at all experimentally accessible spatial scales, it would still be more desirable to *a priori* require the model to accommodate an arbitrarily small lattice spacing, so that in principle, the continuum could then be replicated to arbitrary precision. The remainder of this section describes such a model. However, it should be noted that despite its problems, the previous approach is useful in that it demonstrates how the topology of the lattice is itself a set of parameters, that can be used to alter the dynamics in accordance with the desired equations of motion.

The modulation method

Another way of changing the wave equation into the Klein-Gordon equation is to introduce mass parameters in such a way so as to make any resultant linear perturbations in the associated dispersion relation vanish. With this in mind, let the number of neighboring arcs be doubled. For example, in two dimensions, let it now be the case that there are two arcs connecting a point to any of its nearest neighbors: two arcs in the x_+ direction, two in the y_+ direction and so on. Alternatively, one may suppose that the associated tokens have yet another degree of freedom, and can be in either, say, a 'top' or 'bottom' state. Each one of the doubled number of arcs will—just as before—lead to separate flows, which will be respectively denoted by f_{x_+} and f_{X_+}, etc. The equation of motion for the traveling waves then take on the form

$$\begin{pmatrix} f_{x+} \\ f_{x-} \\ f_{y+} \\ f_{y-} \\ f_{X+} \\ f_{X-} \\ f_{Y+} \\ f_{Y-} \end{pmatrix}^{\text{out}} = \frac{1}{4} \begin{bmatrix} 1 & -3 & 1 & 1 & 1 & 1 & 1 & 1 \\ -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 \\ 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 \\ 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 \\ 1 & 1 & 1 & 1 & 1 & 1 & -3 & 1 \end{bmatrix} \begin{pmatrix} f_{x+} \\ f_{x-} \\ f_{y+} \\ f_{y-} \\ f_{X+} \\ f_{X-} \\ f_{Y+} \\ f_{Y-} \end{pmatrix}^{\text{in}} . \tag{40}$$

The system is still, in effect, two-dimensional. Indeed, one sees that by adding the respective 'top' and 'bottom' coefficients of the matrix as one flow (e.g., ${f'}_{x+} \equiv f_{x+} + f_{X+}$, and so on), the original two-dimensional system of equations (16) is recovered.

Next, divide the above matrix into quadrants and multiply the coefficient in the two off-diagonal quadrants by $e^{-i\alpha}$ and every coefficient in the diagonal quadrants by $e^{i\alpha}$. How this modulation can be implemented by way of a discrete particle-like process is discussed below.

By a simple calculation, one can verify that the system satisfies the relation

$$\frac{\partial^2}{\partial t^2}\psi = c^2(\cos\alpha)\nabla^2\psi - 4(\sin^2\frac{\alpha}{2})\psi \tag{41}$$

where the partial derivatives are as defined in (1) so that they are as yet dimensionless; as before, the terms in the above equation containing factors of $\psi(x, y, t)$ not displaced by one unit of space or time all cancel. If one computes the resultant dispersion relation for this system, one obtains

$$\cos\omega = \frac{\cos\alpha}{2}(\cos\kappa_x + \cos\kappa_y) \tag{42}$$

Note again that as in the case of the wave equations, the frequencies are always real. Expanding the above equation in a power series shows that for a small lattice spacing, and sufficiently small α, this system satisfies a dispersion relation of the form

$$\omega^2 = \frac{1}{2}\left(k_x^2 + k_y^2 + (\alpha')^2\right) , \tag{43}$$

where the k_x and k_y have their usual continuum definitions, and where α' is α divided by the lattice spacing so as to have dimensions of inverse length. This again, is precisely the form of the dispersion relation of the Klein-Gordon equation. Note that by negating α in (40), another system satisfying the same dispersion relation is obtained. This approach can immediately be extended to any number of dimensions and has an additional advantage over the previous one in that the extra components that have been introduced do not change (up to first order) the value of the speed of light from $1/\sqrt{n}$.

For future reference, it will be helpful to introduce some notation. Let $T[m]$ stand for the $2m \times 2m$ matrix, the j^{th} column of which has the property that all its coefficients are $1/m$ except for the single coefficient $T[m]_{jk}$ which is equal to $-(m-1)/m$, where

$$k = \begin{cases} j+1 & \text{if } j \text{ is odd} \\ j-1 & \text{if } j \text{ is even .} \end{cases}$$

Thus the matrices in (16), (18), and (40) are equal to $T[2]$, $T[3]$, and $T[4]$, respectively. A matrix $T[m]$ whose coefficients in its off-diagonal quadrants are multiplied by $e^{-i\alpha}$, while those in its diagonal quadrants are multiplied by $e^{i\alpha}$ will be denoted as $\tilde{T}[m,\alpha]$, so that $\tilde{T}[m,0] = \tilde{T}[m]$. Thus,

$$\tilde{T}[2]\Big|_{\alpha} = \frac{1}{2} \begin{bmatrix} e^{+i\alpha} & -e^{+i\alpha} & e^{-i\alpha} & e^{-i\alpha} \\ -e^{+i\alpha} & e^{+i\alpha} & e^{-i\alpha} & e^{-i\alpha} \\ e^{-i\alpha} & e^{-i\alpha} & e^{+i\alpha} & -e^{+i\alpha} \\ e^{-i\alpha} & e^{-i\alpha} & -e^{+i\alpha} & e^{+i\alpha} \end{bmatrix} . \tag{44}$$

Note that the $\tilde{T}[m,\alpha]$ are also unitary.

Statistical modulation

The lattice interaction whereby an incoming flow is modulated by a factor of $e^{i\alpha}$ from what it would have been in the case of the wave equation, may be implemented via a particle approach, just as previously, the flows associated with the wave equation were interpreted in terms of the statistical motions of discrete particles. It will next be shown that a gas-like population of Poissonly distributed background particles can be used to effect (statistically) the desired modulation.

Consider again the lattice in which the Klein-Gordon equation is to be implemented. Aside from the tokens executing wavelike motion, let there also be a number of particles comprising what will be called the *moderator* gas. The dynamics of these particles as they move from point to point along the arcs is arbitrary, except that it is assumed that they have at all times a Poisson distribution. That is, at each arc, there is a probability

$$P(\alpha;k) = \frac{\alpha^k}{k!}e^{-\alpha} \tag{45}$$

of finding exactly k particles. Therefore, if the number of points in the lattice is N^n, the expected number of moderator tokens, assuming $2n$ arcs per point, is $2n\alpha N^n$.

Suppose next that the presence of one moderator token in an incoming arc multiplies the outgoing distribution by $(1+i\epsilon)$, where ϵ is a positive number which shall be assumed to be less than or equal to one. That is, whereas in the absence of any moderator tokens, the presence of a wave token of phase ϕ produces outgoing wave tokens of the same phase with a probability of $1/4$, now there will also be produced, with a probability of $\epsilon/4$, a wave token of phase $i\phi$. For the reverse arc, where the probability associated with the outgoing tokens is $3/4$ and the phase of the outgoing tokens is $-\phi$, the tokens induced by the moderator particles are likewise produced with a probability of $3\epsilon/4$ and a phase of $-i\phi$. (As always, any resultant tokens of opposite phases annihilate each other.) In the absence of any moderator tokens, there is no extra ϵ interaction.

Thus the moderator tokens act to multiply the number of wave tokens. In a given arc, in the presence of one moderator token, the input of one wave token leads to an expected number of $(1 + i\epsilon)$ tokens. An additional moderator token will act on the output tokens of the first one in the same way, resulting in an expected number of $(1 + i\epsilon)^2$ tokens.

Therefore, the net expected number of output tokens is

$$e^{-\alpha} + (1 + i\epsilon)\frac{\alpha}{1!}e^{-\alpha} + (1 + i\epsilon)^2\frac{\alpha^2}{2!}e^{-\alpha}\ldots = e^{i\epsilon\alpha} \ .$$

Thus, a discrete process is sufficient to modulate the statistical flows of tokens by an arbitrary real parameter. The moderator distribution thus serves here a function similar to that of the Higgs fields of quantum field theories, insofar as it induces a mass. Note that if the extra token which the moderators produced are of a phase $-i$ times the phase of the initial token, making ϵ, in effect, a negative number, then the corresponding modulation factor likewise has a negative phase. The remainder of this section shows that moderator particles that are themselves multiply phased can also be used to induce a modulation factor.

Suppose next that there are two types of moderator tokens, each with a Poisson distribution parametrized by α and α', respectively, and both of which operate on the wave tokens of phase ϕ to produce an additional token of phase $i\phi$, with the probabilities ϵ and ϵ'. The modulation is then

$$e^{i\epsilon\alpha + i\epsilon'\alpha'} \ .$$

Next consider a situation in which the two types of moderator tokens, each initially with a Poisson distribution parametrized by α_+ and α_-, at each arc of the lattice, are given the respective opposite phases of $+1$ and -1. That is, the moderator tokens are themselves given phases, with all the α_+ tokens having a positive phase, and all the α_- tokens having a negative one. (Note that imaginary phases are not included.) Each positive and negative tokens multiplies the outgoing distribution of wave function tokens by a factor of $(1 + i\epsilon)$ and $(1 - i\epsilon)$, respectively. Where moderator tokens of both types are found in any arc of the lattice, they again will be made to annihilate, so that only one type of token remains.

Thus, in considering both types of tokens at once, one can say that at any arc of the lattice, there is a certain probability of finding any number k of tokens, where k may be negative as well as positive. Finding a negative number of tokens simply means finding a positive number of tokens of the second type.

The probability of finding any number is then given by

$$P(\alpha_+, \alpha_-; k) = \begin{cases} \sum_{m=0}^{\infty} P(\alpha_+; k + m)P(\alpha_-; m) & k \geq 0 \\ \sum_{m=0}^{\infty} P(\alpha_+; m)P(\alpha_-; m - k) & k < 0 \end{cases} \ . \tag{46}$$

Writing out the individual Poisson distributions, one obtains

$$P(\alpha_+, \alpha_-; k) = e^{-\alpha_+ - \alpha_-} \sum_{m=0}^{\infty} \frac{{\alpha_+}^{|k|+m} {\alpha_-}^{m}}{(|k|+m)!m!} \tag{47}$$

$$= e^{-\alpha_+ - \alpha_-} \left(\frac{\alpha_+}{\alpha_-}\right)^{k/2} I_{|k|}\left(\sqrt{4\alpha_+\alpha_-}\right)$$

$$k = \ldots, -1, 0, 1, 2, \ldots \ ,$$

where $I_k(x)$ is the modified Bessel function of the first kind, of order k, which satisfies the differential equation

$$x^2 I''(x) + x I'(x) = (x^2 + k^2) I(x) \ .$$

The modulation factor arising from such a distribution of tokens is proportional to $(\alpha_+ - \alpha_-)$.

In the limit of infinitely long times, the same kind of modulation can be produced by a moderator with a binomial instead of a Poisson distribution, because of the similarity of the two when the appropriate limits are taken. (By a binomial distribution, what is meant is that there is a probability p of finding one moderator token at any arc, and a probability $(1-p)$ of finding none.) The ability to statistically modulate quantities by an arbitrary phase shift is useful not only for the simulation of the Klein-Gordon equation, but for the analogous equation for particles subject to electromagnetic or other potentials that themselves satisfy wave equations. The potential amplitude can, just as the quantum wave functions and the moderator, be given statistically with discrete tokens, which will operate on the quantum wave functions just as the moderator tokens do.

7.2 Maxwell's equations

The scalar wave equation considered above can readily be extended into a system for simulating electromagnetic potentials. Initially, it will be assumed that the potentials are generated by classical point charges whose positions at two successive time steps are always on nearest neighboring lattice points—and thus may be viewed as traveling along the arcs of the lattice. These charges will emit tokens in such a way as to yield (on the average) an electromagnetic field obeying the Lorentz gauge, this being the most convenient choice. Moreover, it will be assumed that the charges in question are not themselves affected by the fields they generate—i.e., the tokens they emit.

It is assumed that in addition to being distinguished according to phase, the a token may be in one of four states, labeled the A_x, A_y, A_z and V states—with the distinction depending on the motion of the charges (i.e., the succession of nearest neighboring points on which the charges are located) at the time the tokens were emitted.

The Lorentz gauge condition

$$\frac{1}{c}\frac{\partial}{\partial t}V = -\nabla \cdot \mathbf{A} \tag{48}$$

will be assumed to be valid at any two time steps, on the average. (This is trivially true in the case when currents and fields were initially zero and result from, say, positive-negative charge separation.) Moreover, the conservation of charge will be assumed to hold absolutely, even though situations in which the charges are conserved only on the average are also within the scope of the formalism. The charges shall be assumed to be real-valued, with fixed values of 1 or -1.

Note that (48) can be used as the defining relation for the scalar fields (i.e. the V tokens) in terms of the remaining vector field tokens. The emission rules for the vector field tokens can then be made to correspond in the macroscopic limit to the equations

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = \frac{\mathbf{J}}{c} \ , \tag{49}$$

where the dependence on time and space customarily be suppressed. Note the above three equations are defined on Euclidean space.

It remains to specify the charge and current densities. The value of the charge density at any lattice point will naturally be specified as the net number of charges at the point divided by the appropriate power of the lattice spacing. The current density $\mathbf{J}^{\rm in}(\mathbf{x})$ is defined in terms of the charges on the arcs leading into the point $\mathbf{x}$. A positive charge located at $\mathbf{x}$ which was located at $\mathbf{x} - \mathbf{e}_i$ at the previous time step and which in the subsequent time step will travel to $\mathbf{x} + \mathbf{e}_{i'}$ will contribute an amount of $\frac{1}{2}(\mathbf{J}_i + \mathbf{J}_{i'})$ to the current density. Likewise, negating either $\mathbf{e}_i$ or $\mathbf{e}_{i'}$ (or the charge of the particle in question) will negate the corresponding contribution to the current. With such a definition, the equations (49) will be time-reversal invariant.

Every lattice point containing charges shall emit field tokens. There are in fact several ways of specifying the emission probabilities so that (49) is satisfied; for present purposes, it suffices to assume that the presence at $\mathbf{x}$ of any positive charge which in the previous time step moves along the positive $j^{\rm th}$ axis will generate positive tokens in the two arcs along the axis in question (in the same way as an incoming token would), with a probability of $\eta/2$, where η is a positive number less than or equal to unity. Reversing the charge reverses the phase of the emitted tokens, as does reversing the orientation of the incoming charge. Tokens will also be emitted corresponding to the step which the tokens will make in the subsequent time step, according to the same conventions. Thus, assuming that a positive charge at $\mathbf{x}$ traveled along the positive x in the previous time step, and will travel along the negative y axis in the subsequent time step, positively phased tokens will be emitted in each of the arcs along the x axis, and negatively phased tokens will be emitted in the arcs along the y axis, with the probability of an emitted particle being found in either step being (independently) equal to $\eta/2$. Such an emission probability assures that (49) is satisfied.

Let

$$\frac{\partial}{\partial x_j} f(\mathbf{x}, t) \equiv f(\mathbf{x} + \mathbf{e}_j, t) - f(\mathbf{x} - \mathbf{e}_j, t) \ , \tag{50}$$

where the k^{th} component of $\mathbf{e}_j$ is δ_{jk}, and let the time derivative be similarly defined. For the above emission rules, it may be shown by direct computation that the Lorentz gauge condition (48) will be satisfied if the scalar tokens emitted in the *negative* direction along any axis (i.e., the $x-$, $y-$ or $z-$ direction) are weighted by an extra factor of -1. Thus, whenever a charge emits say positive A_x token along the positive and negative x axis, it will also emit a positive V token along the positive x axis and a negative V token along the negative x axis.

Given that (48) is assumed to be true at two successive time steps, it can then easily be shown that the relation

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} V = \frac{\rho}{c} \tag{51}$$

is satisfied at all times. Moreover, it may be shown that the relations (48) and (49) are preserved through time, by taking the discrete time derivative of both sides of the equations.

7.3 The Dirac equation

The Dirac equation

Solutions to the Dirac equation may also be simulated by the present formalism. Recall that the massless Dirac equation (i.e., the Weyl equation) may be written as

$$\frac{i}{c}\frac{\partial}{\partial t}\begin{bmatrix} \varphi \\ \chi \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{p}\cdot\vec{\sigma} \\ \mathbf{p}\cdot\vec{\sigma} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \varphi \\ \chi \end{bmatrix} \ , \tag{52}$$

where φ and χ are both two-component spinors (whose dependence on space-time has been suppressed), the j^{th} component of $\mathbf{p}$ is $i\frac{\partial}{\partial x_j}$, where the partial derivative is as defined in (50), and the components of the vector $\vec{\sigma}$ are the respective Pauli spin matrices.

It is desirable to retain as much of the formalism of hodotic solutions as possible. The form of the 'transition' matrix in (52), suggests the ansatz that in the case of the Dirac equation, the coefficients transforming the input flows into the output flows should be equal to their scalar-wave versions times an additional generalized phase factor. Recall that in the case of scalar wave equations, the token phase factors formed a cyclic group of four elements. In the case of Dirac equations, a token will be allowed to have an extra two-fold degree of freedom, whereby it may be said to be in either a "spin-up" or "spin-down" state, with the two being distinguished by the usual spinor representation. Also, there will be an additional two-fold degree of freedom whereby a token may be in either a "ϕ" or a "χ" state. The overall generalized phase will then be a Kronecker product of these three factors which shall be referred to as the scalar, spin, and antipodal phases, respectively.

Note also that because of the way in which the discrete analogues of partial derivatives are defined in (50), and the off-diagonal form of the right-hand side of (52), the upper two components of the equation are dependent only on the lower two, and vice versa. A simple change of variables at either the even or odd time steps then allows (52) to be brought into block-diagonal form, as could have been shown by more standard methods. Therefore, one may in fact decrease the number of components in the system of equations by half (though in the present formalism the number of components will again be doubled when mass is incorporated).

The form of the resultant Weyl equation—particularly terms of the form $\mathbf{p} \cdot \vec{\sigma}$—also suggest that the discrete dynamics of the tokens should be modified as follows: let a token that takes a step along the positive (negative) x_j axis be multiplied by a positive (negative) factor of σ_j. Again, the reference to a token taking a step is made by identifying the input and output particle at any lattice point, even though there may be several of the latter for every one of the former.

As an example, a spin-down token with a positive phase, denoted by the spinor $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which takes a step along the positive y axis will in the subsequent time step have the form $\sigma_y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ state (where $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$). That is, it will subsequently be a spin-up token having a phase of $-i$. As before, tokens having the same spin and antipodal phases, but opposite scalar phases will be assumed to annihilate.

It can then be shown by a trivial computation that the above system does indeed satisfy the Weyl equation, where the discrete analogues of the first order partial derivatives are defined as before, and where the speed of light is now $1/n$ instead of $\sqrt{1/n}$. This latter change—which comes about because the (discrete) second order partial derivatives now involve terms displaced by *two* units of space-time instead of just one—ensures that that the frequencies will contain no imaginary component, as before, so that the solutions remain bounded. The introduction of a mass term, transforming the Weyl equation into the Dirac equation, is achieved in much the same way as in the case of the Klein-Gordon equation, i.e., through doubling the number of arcs connecting nearest neighboring points and modulating the corresponding $T[m]$ matrix into $\tilde{T}[2m, \alpha]$.

Explicitly, the equations of motion for the ϕ spinors are of the form

$$\begin{pmatrix} \phi_{x+} \\ \phi_{x-} \\ \phi_{y+} \\ \phi_{y-} \\ \phi_{z+} \\ \phi_{z-} \\ \phi_{X+} \\ \phi_{X-} \\ \phi_{Y+} \\ \phi_{Y-} \\ \phi_{Z+} \\ \phi_{Z-} \end{pmatrix}^{\text{out}} = \text{Diag}(\begin{bmatrix} \sigma_x \\ -\sigma_x \\ \sigma_y \\ -\sigma_y \\ \sigma_z \\ -\sigma_z \\ \sigma_x \\ -\sigma_x \\ \sigma_y \\ -\sigma_y \\ \sigma_z \\ -\sigma_z \end{bmatrix})(\tilde{T}[6,\alpha] \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) \begin{pmatrix} \chi_{x+} \\ \chi_{x-} \\ \chi_{y+} \\ \chi_{y-} \\ \chi_{z+} \\ \chi_{z-} \\ \chi_{X+} \\ \chi_{X-} \\ \chi_{Y+} \\ \chi_{Y-} \\ \chi_{Z+} \\ \chi_{Z-} \end{pmatrix}^{\text{in}}, \tag{53}$$

where the 'ϕ_{x_+}' and 'ϕ_{X_+}' refer to the two (spinor) flows of the ϕ variable exiting the point in question along the positive x axis, and so on. Moreover, the Kronecker product operation implies that each coefficient in $\tilde{T}[6,\alpha]$ multiplies a 2×2 identity matrix that acts on spinors, and the first entry on the right hand side stands for the bock diagonal matrix (with blocks having a dimensionality of 2×2) whose nonzero coefficients are Pauli matrices, as indicated. As in the case of the Klein-Gordon equations, the modulation imposed by the exponential factors may be implemented statistically, by way of a background lattice gas of Poissonly distributed moderator tokens.

Let a similar set of equations apply to the χ spinors, except that the variable α is everywhere negated (in keeping with the usu al interpretation that the ϕ and χ spinors represent the positive and negative mass states). It may then be verified by simple algebra that the equations of motion satisfy

$$\frac{\partial t^2}{\partial t^2}\psi = c^2 \cos^2(\alpha)\nabla^2\psi - 4\sin^2(\alpha)\psi \tag{54}$$

where ψ stands for the four-component spinor whose top and bottom components are the ϕ and χ spinors respectively. In keeping with the above definition for partial derivatives (50), the components of the Laplacian here satisfy the relation

$$\frac{\partial^2}{\partial {x_i}^2} f(\mathbf{x}, t) = f(\ldots, x_i + 2, \ldots) - 2f(\ldots, x_i, \ldots) + f(\ldots, x_i - 2, \ldots) \tag{55}$$

and likewise for the derivative involving time. By expanding the trigonometric terms depending on α in a power series about zero, it may be shown that for sufficiently small values of α, the dispersion equation of this system takes on its desired Klein-Gordon form, with α again serving as the mass (up to a factor involving the lattice spacing). Had the $\tilde{T}[6,\alpha]$ matrix in (53) and its analogue for χ been identical (as opposed to complex conjugates) then the resultant second order equations would have been equivalent to the wave equation, except that the speed of light would have in effect been multiplied by a factor of $\cos\alpha$. By properly generalizing the σ_j matrices, one may correspondingly apply the present formalism to any number of dimensions. In each case, the coefficients of the associated transition matrices are multiplied by elements of a group corresponding to the internal structure of the particles in question. Parameters such as mass, which have the effect of modulating the flow of tokens by some complex phase factor, are manifested as the perturbations induced by a Poissonly distributed field of tokens. Other fundamental equations of particle physics can likewise be simulated by similar means.

The equations given here involving the point solutions have the property that they are correct regardless of whether the solutions at a point are defined to be sums of incoming flows or outgoing flows. If, say, only the sum of outgoing flows is to be used, then there will exist several other transition matrices that yield (54), which are nontrivial modifications of the ones given here. Note again that while the number of components appears to have been doubled, the antipodal phase can be removed by a simple change of variables. Moreover, as before, the hodotic solutions for this system may be shown to be (over)complete, by constructing Green's functions expressed in terms of interlocking hodotic solutions.

In the case where $\alpha = 0$, the present formalism can of course be used to simulate the scalar wave equation, and by extension, the Maxwell's equations considered previously. However, the speed of light in such a case will be equal to $1/n$ instead of $\sqrt{1/n}$.

To incorporate potentials into this formalism, corresponding to those of a classical background field, consider modulating every wave function token entering a point $\mathbf{x}$ by an amount $e^{i\Delta^{\text{in}}(\mathbf{x})/2}$ where

$$\Delta^{\text{in}}(\mathbf{x}) = V^{\text{in}}(\mathbf{x}) - \mathbf{A}^{\text{in}}(\mathbf{x}) \cdot u^{\text{in}} \tag{56}$$

where u is a unit vector pointed along the arc on which the token in question is traveling; the superscripts refer to to the fact that all the variables have values dependent on the incoming field tokens. If $\Delta^{\text{out}}(\mathbf{x})$ is analogously defined in terms of the tokens exiting the point $\mathbf{x}$, and the exiting wave function tokens are further modulated by a factor of $e^{i\Delta^{\text{out}}(\mathbf{x})/2}$, the system may be shown to converge to the Dirac equation interacting with a classical field. Alternate treatments, in which the incorporation of the potentials is carried out only in terms of incoming or exiting tokens can also be considered, at the cost of forsaking time-reversal symmetry (of the expected values of the amplitudes). A self-consistent system of charged particles and fields, in which the only fields present are those the (quantum mechanical) particles themselves emit, may be implemented by expressing the currents in terms of the bra and ket occupation numbers $\check{n}(\mathbf{x}, t)\hat{n}(\mathbf{x}, t)$.

7.4 Hodotic solutions

In reference to the first (i.e., scalar) wave equations considered here, some unusual properties of its associated hodotic solutions are easily established by resorting to summations over paths. As before, let $\sum_{\mathbf{x}} f(\mathbf{x}, t)$, be the sum over all space of some function $f(\mathbf{x}, t)$ at some instant of time. That is, it is a sum over the *point* values of a given solution.

It is easy to show from the equations of motion that the hodotic solutions have the property that $\sum_{\mathbf{x}} h_\sigma(\mathbf{x}, t)$ is equal to one for all time, where as before,

$$\sigma \in \{x_{1+}, x_{1-}, x_{2+}, \ldots, x_{n-}\} \quad .$$

What is much more remarkable is that on any n-dimensional torus whose lengths along any of the coordinate axes are the same, the quantity $\sum_{\mathbf{x}} h_\sigma^2(\mathbf{x}, t)$ is also equal to one for all time. That is, without regard to the traveling wave *arc* solutions, the integral of the square of the *point* solution is conserved. (This property does not hold, in general, for an arbitrary sum of the hodotic solutions.) The proof of this conservation is somewhat tedious, but it will be outlined briefly, using the two-dimensional case as an example.

As noted above, $h_{x+}(x, y, T)$ may be written as a sum over all walks of length T that start the origin, go immediately afterwards to the point $(1, 0)$, then on to (x, y). By the same argument, $h_{x+}^2(x, y, T)$ may also be written as a sum of terms, each of which now represents a *pair* of such paths. Explicitly,

$$h_{x+}^2(x, y, T) = \sum_{I_{x+}, I'_{x+}} (1/2)^{2T} (-1)^{R(I_{x+}) + R(I'_{x+})} \tag{57}$$

where I'_{x+} is an indexing of paths identical to I_{x+}, and the rest of the notation is again the same as for (27) and (28).

Let $h_{x+}^\sigma(x, y, T)$ represent the contribution to $h_{x+}(x, y, T)$ of paths whose T^{th} (i.e., whose final) step is along the x_+ direction, so that

$$h_{x+}(x, y, T) = \sum_\sigma h_{x+}^\sigma(x, y, T) \ . \tag{58}$$

Then,

$$\sum_{\mathbf{x}} \left(h_{x+}\right)^2 = \sum_{\mathbf{x}} \sum_\sigma \left(h_{x+}^\sigma\right)^2 + \sum_{\mathbf{x}} \sum_{\sigma \neq \sigma'} h_{x+}^\sigma h_{x+}^{\sigma'} \ , \tag{59}$$

where the functional dependence on x, y, and T has been suppressed. Now the first (double) sum on the right is equal to one. This is because it represents the discrete integral of the sum of the squares of the traveling wave components for this solution of the wave equation. This quantity, due to the unitarity of the transition matrix, is always conserved (and therefore equal to its initial value of one). Therefore, it remains only to show that the second sum on the right vanishes.

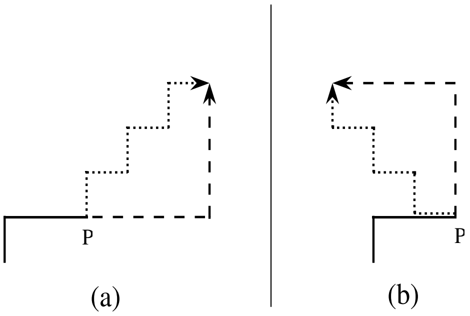

Figure 8: Two paths (a) that are identical except for their final m steps, where $m \neq 0$, but arrive at the same place at $t = T$, may by a symmetry operation be related to another pair of paths (b) such that the total number of path reversals in the latter pair differs by one, so that the net contribution of these two pairs is zero.

First suppose that the two-dimensional torus under consideration is of infinite length along either axis. Unlike the first sum on the right-hand side, the second contains no terms representing pairs of walks that are identical. This is because if the two paths corresponding to such a term were identical, they would obviously have their T^{th} step along the same direction and therefore would belong in the first sum. It is easy to see graphically, that any such pair of nonidentical paths may be related in a one-to-one way to another pair of paths (belonging to the same summation) whose cumulative number of reversals differs by one (Figure 8). Since the value of the contribution of any pair of paths depends only on the number of reversals they contain, the contributions to the summation from these two sets of paths therefore cancel one another (Figure 8). (In fact, one can extend this argument to show that (59) would be unchanged if the right-hand side summands were multiplied by $\delta_{I,I'}$.)

The same result can be obtained without resorting to graphical methods by representing each path as a string of T choices from the set of possible directions, and then showing that the contributions from certain classes of strings cancel.

To extend these results to a torus, it is also necessary to consider terms in the sum representing *wrapping pairs,* i.e., pairs of paths that when connected end to end describe a loop that is not homotopic to a point, but instead wraps around the torus along some direction. (The reference to homotopy is, of course, made with regard to the continuous paths that can be obtained from the discrete ones by imbedding the lattice in a Euclidean space, and by likewise transforming any step between two lattice points into a continuous path along the straight line segment joining those points.)

If the two axes of the torus are of equal length, then the contributions from the pairs of paths that wrap around the torus along one direction may be seen to cancel the contributions of those that wrap in the perpendicular direction. However, suppose

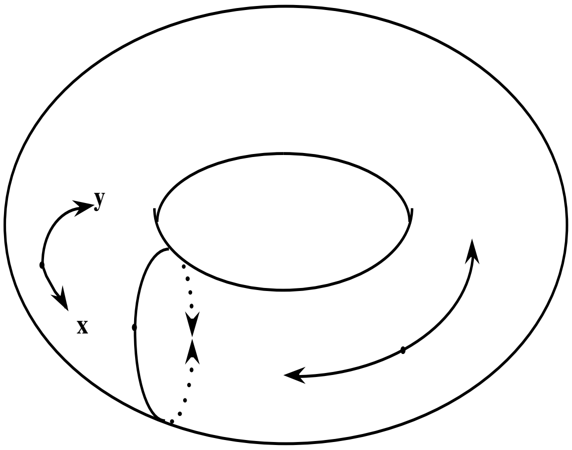

Figure 9: If the two axes of the torus are of unequal length, say, $L_y > L_x$, then at $t = L_x/2 + 1 \equiv T$, a pair of paths forming a loop along the x-axis will not be cancelled by a corresponding loop along the y-direction, since such loops could only be made of paths of length greater than T.

the lengths of the lattice along the two axes are unequal. Then for sufficiently large T, there will be a pair of paths wrapping around the shorter length of the torus whose contribution to the integral will not be cancelled by a pair of paths wrapping along the other axis, because the latter axis is too long to traverse in T time steps (Figure 9).

In $n > 2$ dimensions, the proof is more tedious, because the coefficients in the transition matrix no longer have the same modulus. Let a pair of nonidentical paths be called reversal-diverging or *trans*versal-diverging, respectively according to whether or not the two paths diverge by having one of the paths execute a reverse step (Figure 10). (Remember, all of the paths under consideration already have an identical first step.) The contribution of a pair of reversal-diverging paths must then be added to the contributions of $(n-1)$ pairs of transversal-diverging paths in order to obtain a cancellation. In extending the proof to an n-dimensional torus, one will similarly have to take special notice of the wrapping pairs, which again may be classed according to whether they are reversal-diverging or transversal-diverging. The contribution from such a pair of reversal-diverging paths will cancel out the contributions from $(n-1)$ pairs of transversal-diverging paths, but again, only if the axes of the torus are of equal length.

We turn next to another curious property of the hodotic solutions, that becomes

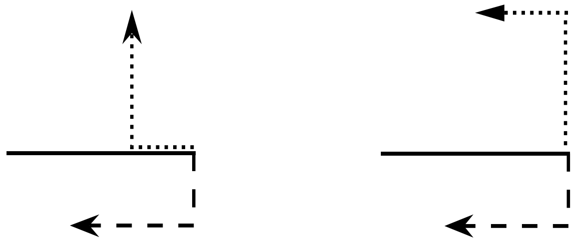

Figure 10: The pair of paths on the left diverges with one path making a reversal; it is said to be reversal-diverging. The pair on the right diverges with neither of the paths making a reversal; it is said to be transversal-diverging. (The solid line indicates the portions of the two paths that are identical.)

useful in decomposing solutions of the wave equation into components other than the $2n$ traveling waves discussed above. Let *promotion* of an m-dimensional solution of (2) to an n-dimensional space, with $n > m$, denote the following procedure:

> take the (m-dimensional) hodotic solution at any two successive time values and imbed it in a space of dimension n. That is, use the solution as the boundary conditions on a hyperplane of dimension m in an n-dimensional space, with the solution being zero everywhere outside the hyperplane. Afterwards, iterate according to (2), (the speed of light becoming $\sqrt{1/n}$).

The hodotic solutions have the interesting property that they may be iterated an arbitrary number of times, promoted to a higher dimension, iterated again for some other arbitrary number of times, promoted to a still higherdimensional space, and so on, all the while producing a solution whose discrete integrals over the corresponding space of the point solution, as well as its square, are constant. The proof is similar to the one outlined previously; classes of pairs of nonidentical paths can again be shown to cancel, although for the purposes of the proof, the number of classes to which a pair of paths may belong varies according to the time at which the two members of the pair diverged (more precisely, according to whether they diverged before or after the given imbedding in the higher dimensional space). Note that in general, this result does not hold for a torus.

Finally, note that the n-dimensional hodotic solutions, are simply the promotions of one-dimensional hodotic solutions to the appropriate higher dimension.

Acknowledgments

I wish to thank Tommaso Toffoli, Norman Margolus, and all the members of the Information Mechanics Group at MIT, whose help made this work possible. Support was provided in part by the National Science Foundation, grant no. 8618002-IRI, and in part by the Defense Advanced Research Projects Agency, grant no. N00014-89-J-1988.

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