The maximum speed of dynamical evolution

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Abstract

We discuss the problem of counting the maximum number of distinct states that an isolated physical system can pass through in a given period of time—its *maximum speed of dynamical evolution*. Previous analyses have given bounds in terms of ΔE, the standard deviation of the energy of the system; here we give a strict bound that depends only on $E - E_0$, the system's average energy minus its ground state energy. We also discuss bounds on information processing rates implied by our bound on the speed of dynamical evolution. For example, adding one Joule of energy to a given computer can never increase its processing rate by more than about 3×10^{33} operations per second.

1 Introduction

In the realm of computation, the first two quantitative questions that one is likely to ask about a machine are *(i) How much memory does it have?* and *(ii) How fast does it run?* In exploring the computational limits of physical dynamics, one might try to ask the same questions about an arbitrary physical system.

Question *(i)* essentially asks, "How many distinct states can my system be put into, subject to whatever physical constraints I have?" This is really a very old question: the correct counting of physical states is the problem that led to the introduction of Planck's constant into physics[11], and is the basis of all of

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quantum statistical mechanics. This question can be answered by a detailed quantum mechanical counting of distinct (mutually orthogonal) states. It can also be well approximated in the macroscopic limit[9,21] by simply computing the volume of phase space accessible to the system, in units where Planck's constant is 1.

Question *(ii)* will be the focus of this paper. This question can be asked with various levels of sophistication. Here we will discuss a particularly simple measure of speed: the maximum number of distinct states that the system can pass through, per unit of time. For a classical computer, this would correspond to the maximum number of operations per second.

For a quantum system, the notion of distinct states is well defined: two states are distinct if they are orthogonal. The connection between orthogonality and rate of information processing has previously been discussed[12,2,7,13,14,4], but no universal bound on computation rate was proposed. The minimum time needed for a quantum system to pass from one orthogonal state to another has also previously been characterized, in terms of the standard deviation of the energy ΔE[10,16,20,17,5]. This bound places no limit, however, on how fast a system with bounded average energy can evolve (since ΔE can be arbitrarily large with fixed E). Bounds based directly on the average energy E have previously been proposed[6,3], but these bounds apply to the rate of communication of bits, rather than to the rate of orthogonal evolution: difficulties associated with such bit-related bounds are discussed in [12]. The new bounds derived in this paper are also based on average energy, but they apply to rates of orthogonal evolution. For an ordinary macroscopic system, these bounds are achievable: we show that adding energy increases the maximum rate at which such a system can pass through a sequence of mutually orthogonal states by a proportionate amount.

There has recently been much interest in the possibilities of *quantum computers*: computers that can operate on superpositions of computational states[19]. Even isolated quantum computers will, in general, pass through sequences of (nearly) mutually orthogonal states in the course of their complicated time evolutions. At the least, an efficient quantum computation should, for some initial states, have a final state that is reasonably distinct from its initial state. Thus a bound on the rate of orthogonal evolution is relevant in this case as well.

2 Maximum rate of dynamics

In the energy basis, quantum time evolutions are constructed out of superpositions of frequency components. One might expect from this that, given a maximum energy eigenvalue, the frequency with which states can change should be bounded by

$$\nu_\perp \leq \frac{E_{\max}}{h} \tag{1}$$

If we take our zero of energy at the ground state of the system,[2] and consider long evolutions, then this relation is true, as we will discuss below. We will also show that, given a fixed average energy E (rather than a fixed maximum), there is a similar bound

$$\nu_\perp \leq \frac{2E}{h}, \tag{2}$$

where again we have taken our zero of energy at the ground state. This equation has the following interpretation: in appropriate units, the average energy of a macroscopic system is equal to the maximum number of orthogonal states that the system can pass through per unit of time. This is the maximum rate that can be sustained for a long evolution—the rate at which a system can oscillate between two states is twice as great.

2.1 Orthogonality time

We begin our analysis by discussing the question of the minimum time needed for *any* state of a given physical system to evolve into an orthogonal state.

An arbitrary quantum state can be written as a superposition of energy eigenstates

$$|\psi_0\rangle = \sum_n c_n |E_n\rangle \tag{3}$$

Here and throughout this paper we assume that our system has a discrete spectrum, and that the states have been numbered so that the energy eigenvalues $\{E_n\}$ associated with the states $\{|E_n\rangle\}$ are non-decreasing. To simplify formulas, we will choose our zero of energy so that $E_0 = 0$.

[2] In all of this discussion, if for some reason no energies below some $E_{\min}$ are allowed, we should take our zero of energy there instead.

Let $\tau_\perp$ be the time it takes for $|\psi_0\rangle$ to evolve into an orthogonal state. We will now show that, with a fixed average energy E, it is always true that

$$\tau_\perp \geq \frac{h}{4E}. \tag{4}$$

This result is somewhat surprising, since earlier results gave a bound only in terms of ΔE

$$\tau_\perp \geq \frac{h}{4\Delta E} \tag{5}$$

This earlier bound would suggest that, given a fixed average energy, one could construct a state with a very large ΔE in order to achieve an arbitrarily short $\tau_\perp$. We will show that this is not the case.

Let us begin by observing that if $|\psi_0\rangle$ is evolved for a time t it becomes

$$|\psi_t\rangle = \sum_n c_n e^{-i\frac{E_n t}{\hbar}} |E_n\rangle \tag{6}$$

Now we let

$$S(t) = \langle\psi_0|\psi_t\rangle = \sum_{n=0}^{\infty} |c_n|^2 e^{-i\frac{E_n t}{\hbar}} \tag{7}$$

We want to find the smallest value of t such that $S(t)$=0. To do this, we note that

$$\begin{aligned}
\mathrm{Re}(S) &= \sum_{n=0}^{\infty} |c_n|^2 \cos(\frac{E_n t}{\hbar}) \\
&\geq \sum_{n=0}^{\infty} |c_n|^2 \left(1 - \frac{2}{\pi}(\frac{E_n t}{\hbar} + \sin(\frac{E_n t}{\hbar}))\right) \\
&= 1 - \frac{2E}{\pi\hbar}t + \frac{2}{\pi}\mathrm{Im}(S),
\end{aligned} \tag{8}$$

where we have used the inequality $\cos x \geq 1 - \frac{2}{\pi}(x + \sin x)$, valid for $x \geq 0$. But for any value of t for which $S(t) = 0$, both $\mathrm{Re}(S) = 0$ and $\mathrm{Im}(S) = 0$, and so Eq. (8) becomes

$$0 \geq 1 - \frac{4Et}{h} \tag{9}$$

Thus the earliest that $S(t)$ can possibly equal zero is when $t = h/4E$, which proves Eq. (4). Of course Eq. (8) also gives approximately the same bound on how quickly we can have approximate orthogonality, since if $|S(t)|$ is small, then so are $\mathrm{Re}(S)$ and $\mathrm{Im}(S)$.

This bound is achievable if the spectrum of energies includes the energy $2E$ (and is very nearly achievable if the spectrum includes a value very close to this, as we would expect, for example, for any ordinary macroscopic system). In this case, we let

$$|\psi_0\rangle = \frac{|0\rangle + |2E\rangle}{\sqrt{2}} \tag{10}$$

which has average energy E. This evolves in a time $t = h/4E$ into

$$|\psi_t\rangle = \frac{|0\rangle - |2E\rangle}{\sqrt{2}} \tag{11}$$

which is orthogonal to $|\psi_0\rangle$. If we evolve for the same interval again, we will be back to $|\psi_0\rangle$: the evolution oscillates between these two orthogonal states. For these states, $\Delta E = E$, and so both of the bounds Eq. (4) and Eq. (5) are achieved.

There are also cases where Eq. (4) gives a much better bound than Eq. (5). Consider, for example, the state

$$|\psi_0\rangle \quad = \quad a\left(|0\rangle + |\varepsilon\rangle\right) \quad + \quad b\left(|n\varepsilon\rangle + |(n+1)\varepsilon\rangle\right) \tag{12}$$

which evolves into an orthogonal state in a time $\tau_\perp = h/2\varepsilon$. Given E, as long as we choose $\varepsilon < 2E$ (i.e., $\tau_\perp > h/4E$) the average energy of the first pair of kets will be less than E. Given ε, for large enough n the average energy of the second pair of kets will be greater than E. Then we can always find coefficients a and b that make the average energy of $|\psi_0\rangle$ be E and also normalize the state. But this state has a ΔE that depends on our choice of n: in fact $\Delta E = \Theta(\sqrt{n})$. With fixed E, ΔE can be as large as we like. Thus in this case, Eq. (5) is not a useful bound and Eq. (4) is optimal.

2.2 Cycles of orthogonal states

In the discussion above, we have seen that a quantum system with average energy E can be made to oscillate between two orthogonal states with a frequency of $4E/h$. Now we address the question of how fast a quantum system can run through a long sequence of mutually orthogonal states. We begin by considering the very restricted set of evolutions that pass through an exact cycle of N mutually orthogonal states at a constant rate. In this case it is easy to show (see Appendix A) that

$$\tau_\perp \geq \frac{N-1}{N}\frac{h}{2E} \tag{13}$$

Thus for very long evolutions that form a closed cycle, the maximum transition rate between orthogonal states is only half as great as it is for an oscillation between two states. In the next section, we will show that this long-sequence asymptotic rate is achievable in principle for any ordinary macroscopic system. Here we will first give an example of a system for which an *exact* cycle of N mutually orthogonal states (*cf.* [15]) achieves this bound.

The one-dimensional harmonic oscillator has an exact cycle after some period τ. Taking our ground-state energy to be zero, all of the energy eigenvalues are multiples of $\varepsilon_1 = h/\tau$. Let

$$|\psi_0\rangle = \sum_{n=0}^{N-1} \frac{1}{\sqrt{N}} |n\varepsilon_1\rangle \tag{14}$$

If our system passes through N mutually orthogonal states in time τ, then the average time to pass between consecutive orthogonal states is $\tau_\perp = \tau/N$. Noting that $\varepsilon_1 \tau_\perp/\hbar = 2\pi/N$, we see that the state obtained from $|\psi_0\rangle$ after m time intervals of length $\tau_\perp$ is

$$|\psi_m\rangle = \sum_{n=0}^{N-1} \frac{1}{\sqrt{N}} e^{-\frac{2\pi i n m}{N}} |n\varepsilon_1\rangle \tag{15}$$

and so

$$\langle\psi_{m'}|\psi_m\rangle = \sum_{n=0}^{N-1} \frac{1}{N} e^{\frac{2\pi i n}{N}(m'-m)} = \delta_{m'm} \tag{16}$$

Now we can calculate the relationship between E and $\tau_\perp$.

$$\langle\psi_0|H|\psi_0\rangle = \varepsilon_1 \sum_{n=0}^{N-1} \frac{n}{N} = \varepsilon_1 \left(\frac{N-1}{2}\right) = \left(\frac{h}{N\tau_\perp}\right)\left(\frac{N-1}{2}\right) \tag{17}$$

and so

$$\tau_\perp = \frac{N-1}{N}\frac{h}{2E} \tag{18}$$

2.3 Long sequences of orthogonal states

Now we turn to the question of whether ordinary macroscopic physical systems can also run through long sequences of mutually orthogonal states with $\tau_\perp = h/2E$. We will show by construction that they can. As in the discussion above, we will not need to use arbitrarily large eigenvalues to achieve this rate, and so our state can be written

$$|\psi_0\rangle = \sum_{n=0}^{N-1} c_n |E_n\rangle \tag{19}$$

Now we simply let

$$c_n = \sqrt{\frac{E_{n+1} - E_n}{E_N}}. \tag{20}$$

This definition of c_n generalizes our example from the previous section: for the special case of $E_n = n\varepsilon_1$, Eq. (19) reduces to Eq. (14), which achieves $\tau_\perp = h/2E$ in the macroscopic limit. Notice also that, with this definition of c_n, states with degenerate energy eigenvalues are not repeated in our superposition (they get a coefficient of zero, since the E_n's are numbered in non-decreasing order). This definition of c_n always gives normalized states, since

$$\langle\psi_0|\psi_0\rangle = \sum_{n=0}^{N-1} \frac{E_{n+1} - E_n}{E_N} = 1 \tag{21}$$

We can calculate the average energy in the state $|\psi_0\rangle$. This is just

$$\langle\psi_0|H|\psi_0\rangle = \sum_{n,n'} c^*_{n'} c_n E_n \langle E_{n'}|E_n\rangle = \sum_{n=0}^{N-1} \frac{E_{n+1} - E_n}{E_N} E_n \tag{22}$$

For $N \gg 1$ and $c_n \ll 1$, we can approximate this sum by an integral. Letting $x = n/N$ and $\varepsilon(x) = E_n/E_N$, we have

$$\langle\psi_0|H|\psi_0\rangle \approx E_N \int_0^1 \varepsilon \frac{d\varepsilon}{dx} dx = \frac{E_N}{2} \int_0^1 \frac{d}{dx}\varepsilon^2 dx = \frac{E_N}{2} \tag{23}$$

In Appendix B we estimate the corrections to this approximation, which vanish for large N. Thus, with this definition of c_n, by giving equal weight to equal energy intervals we get an average energy that is half of the maximum energy, just as in the limiting case considered in Eq. (17).

Now the state obtained from $|\psi_0\rangle$ after m intervals of length $\tau_\perp = h/2E = h/E_N$ is

$$|\psi_m\rangle = \sum_n c_n e^{-2\pi i m \frac{E_n}{E_N}} |E_n\rangle \tag{24}$$

and so

$$\begin{aligned}\langle\psi_{m'}|\psi_m\rangle &= \sum_{n,n'} c^*_{n'} c_n e^{2\pi i \frac{1}{E_N}(m' E_{n'} - m E_n)} \langle E_{n'}|E_n\rangle \\ &= \sum_{n=0}^{N-1} \frac{E_{n+1} - E_n}{E_N} e^{2\pi i \frac{E_n}{E_N}(m'-m)}\end{aligned} \tag{25}$$

As we've seen, this is equal to 1 if $m' = m$. Let $k = m' - m \neq 0$, and again let $x = n/N$ and $\varepsilon(x) = E_n/E_N$. Then

$$\langle \psi_{m'} | \psi_m \rangle \approx \int_0^1 \frac{d\varepsilon}{dx} e^{2\pi i k \varepsilon} dx = \int_0^1 \frac{\frac{d}{dx} e^{2\pi i k \varepsilon}}{2\pi i k} dx = 0 \tag{26}$$

A more careful analysis (see Appendix B) verifies that the corrections to this approximate calculation vanish for large N. Thus we can run through a long sequence of nearly orthogonal states at the rate $\nu_\perp = 2E/h = E_{\max}/h$.

3 Interpretation

For an isolated macroscopic system s with average energy $E^{(s)}$, we have seen that we can construct a state that evolves at a rate $\nu_\perp^{(s)} = 2E^{(s)}/h$. If we had many non-interacting macroscopic subsystems, we would have an average energy for the combined system of $E_{\text{tot}} = \sum_s E^{(s)}$. Our construction of the previous section applies prefectly well to such a composite system, and in particular lets us construct a state for this combination of non-interacting subsystems that evolves at a rate of $\nu_\perp = 2E_{\text{tot}}/h = \sum_s 2E^{(s)}/h = \sum_s \nu_\perp^{(s)}$. Thus if we subdivide our total energy between separate subsystems, the maximum number of orthogonal states per unit time for the combined system is just the sum of the maximum number for each subsystem taken separately. This is analogous to the case in a parallel computer, where the total number of operations per second for the whole machine is just the sum of the number of operations per second performed by the various pieces. Our result should be interpreted in a similar manner: average energy tells us the maximum possible rate at which distinct changes can occur in our system.

It is interesting to ask how this connection between energy and maximum possible number of changes looks in a semi-classical limit. As a simple example, let us consider a single-speed lattice gas[8]. This is a classical gas model in which identical particles are distributed on a regular lattice. Each particle moves from one lattice site to an adjacent site in time δT. At the end of each δT interval, all energy is kinetic, and all particles have the same energy δE. Thus if the total energy is E, then $E/\delta E$ is equal to the number of particles, and so the maximum number of spots that can change per unit of time is equal to $2E/\delta E$: $E/\delta E$ spots can be vacated, and $E/\delta E$ new spots occupied. Fewer spots will change if some particles move to spots that were previously occupied, but we can never have more than $2E/\delta E$ changes in time δT. Thus if we impose the constraint on this lattice system that $\delta E \delta T \geq h$, our bound on the rate at which spots can change becomes $\nu_{\text{change}} \leq 2E/\delta E \delta T \leq 2E/h$.

It is also interesting to ask how this connection between energy and orthogonal

evolution looks in different inertial reference frames. Clearly we will see some orthogonal states that are purely an artifact of our choice of reference frame. For example, an isolated stationary atom in an exact energy eigenstate never transitions to an orthogonal state, but if we view this same atom from a moving frame we will see a sequence of distinct position states. If we are interested in a bound on "useful" dynamics (e.g., on computation), then we shouldn't count extra states that arise just from the state of motion of the observer. The obvious solution is to define the amount of "useful" dynamics to be the number of orthogonal states that the system passes through in its rest frame (center of momentum frame). As long as our non-relativistic analysis is valid in the rest frame, we can infer that (in that frame) a system with total relativistic energy E_r cannot pass through more than $2E_r t_r/h$ different orthogonal states in time t_r. Then in any frame, we can compute this bound on "useful" evolution, since $E_r t_r = p_\mu x^\mu$. In a frame in which the system starts at the origin at time $t = 0$ and moves in the positive x direction with a momentum of magnitude p, our bound is $(2/h)(Et - px)$: from the time component of the bound we subtract a space component. Note that we subtract one orthogonal state for each shift of a distance $h/2p$ (*cf.* [5,22]).

4 Conclusion

The average energy E (above the ground state) of an isolated physical system tells us directly the maximum number of mutually orthogonal states that the system can pass through per unit of time. Thus if the system is a computer, this quantity is a simple physical measure of its maximum possible *processing rate*.

Equivalently we can say that Et counts orthogonal states in time. Just as accessible phase-space volume tells us the number of distinct states that a macroscopic physical system can be put into, Et "action volume" tells us the number of distinct states that a system with energy E can pass through in time t.

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A Minimum orthogonality time for exact cycles

If we demand that the evolution should *exactly* cycle after N steps, this condition puts a severe restriction on the energy eigenfunctions that can contribute to our initial state. If $|E_j\rangle$ contributes to $|\psi_0\rangle$, and if our cycle length is τ, then we must have that $E_j\tau/\hbar = 2\pi k_j$ for some integer k_j, which means that $E_j = k_j\varepsilon_1$, where $\varepsilon_1 = h/\tau$. Thus our initial state must have the form

$$|\psi_0\rangle = \sum_{n=0}^{\infty} c_n |n\varepsilon_1\rangle \tag{A.1}$$

For simplicity, we have not included degenerate energy eigenfunctions in our superposition—adding these would not affect our conclusions. Thus the assumption of exact periodicity restricts us to systems with energy eigenvalues that are a subset of a one dimensional harmonic oscillator spectrum, in which all energies are multiples of h/τ.

As in Section 2.2, we see that the state obtained from $|\psi_0\rangle$ after m time intervals of length $\tau_\perp = \tau/N$ is

$$|\psi_m\rangle = \sum_{n=0}^{\infty} c_n e^{-\frac{2\pi i n m}{N}} |n\varepsilon_1\rangle \tag{A.2}$$

and so

$$\langle\psi_{m'}|\psi_m\rangle = \sum_{n=0}^{\infty} |c_n|^2 e^{\frac{2\pi i n}{N}(m'-m)} \tag{A.3}$$

There are actually only N distinct values of the exponential that appears in this sum, since for any integer l, $e^{\frac{2\pi i(n+lN)}{N}(m'-m)} = e^{\frac{2\pi i n}{N}(m'-m)}$. We can collect together all c_n's that multiply each distinct value: let $|d_n|^2 = \sum_{l=0}^{\infty} |c_{n+lN}|^2$. Then

$$\langle\psi_{m'}|\psi_m\rangle = \sum_{n=0}^{N-1} |d_n|^2 e^{\frac{2\pi i n}{N}(m'-m)} = \delta_{m'm} \tag{A.4}$$

since these states are supposed to be orthogonal. This last equality will obviously be true if we let $|d_n|^2 = 1/N$ for all n's. In fact, since there are N different possible values of $m' - m$, Eq. (A.4) constitutes N linearly independent equations with N unknown coefficients, and so this solution is unique.

Thus by picking any set of $|c_n|^2$'s that add up to make all the $|d_n|^2$'s equal to $1/N$, we obtain a state $|\psi_0\rangle$ that evolves at a constant rate through a sequence of N mutually orthogonal states in time τ.

Now we can calculate the relationship between the average energy and the orthogonality time.

$$E = \varepsilon_1 \sum_{n=0}^{\infty} |c_n|^2 n = \varepsilon_1 \sum_{n=0}^{N-1} \sum_{l=0}^{\infty} |c_{n+lN}|^2 (n + lN) \tag{A.5}$$

but $\sum_{l=0}^{\infty} |c_{n+lN}|^2 (n + lN) \geq \sum_{l=0}^{\infty} |c_{n+lN}|^2 n = n|d_n|^2$, and since $|d_n|^2 = 1/N$,

$$E \geq \varepsilon_1 \sum_{n=0}^{N-1} \frac{n}{N} = \varepsilon_1 \left(\frac{N-1}{2} \right) \tag{A.6}$$

Since $\varepsilon_1 = h/\tau = h/N\tau_\perp$, this establishes Eq. (13). Note that we get equality in Eq. (13) only for the state given in Eq. (14).

We can also argue that if we don't use energy eigenvalues larger than $E_{\max} = \varepsilon_1(N-1)$, then there are at most N eigenstates in our superposition, and so we can pass through at most N different orthogonal states in time τ. Thus

$$\tau_\perp \geq \frac{\tau}{N} = \frac{h}{\varepsilon_1 N} = \left(\frac{h}{E_{\max}} \right) \left(\frac{N-1}{N} \right) \tag{A.7}$$

which yields Eq. (1) for a long exact cycle.

B Approximating sums

Here we estimate the corrections to the average energy and scalar product computed in Section 2.3.

Letting $\varepsilon_n = E_n/E_N$ and $\delta_n = \varepsilon_{n+1} - \varepsilon_n$, Eq. (22) becomes

$$\begin{aligned} \langle \psi_0 | H | \psi_0 \rangle &= E_N \sum_{n=0}^{N-1} (\varepsilon_{n+1} - \varepsilon_n) \varepsilon_n \\ &= \frac{E_N}{2} \sum_{n=0}^{N-1} \left[(\varepsilon_{n+1}^2 - \varepsilon_n^2) - (\varepsilon_{n+1} - \varepsilon_n)^2 \right] \\ &= \frac{E_N}{2} \left[1 - \sum_{n=0}^{N-1} \delta_n^2 \right] \end{aligned} \tag{B.1}$$

Now, for any ordinary macroscopic system, from general properties of the density of states[18] we know that for large n, $E_n \sim n^c$, where c is a positive constant much less that one. From this we can show that $\sum \delta_n^2 = O(1/N^{2c})$.

Similarly, letting $\alpha = 2\pi(m' - m) = 2\pi k$, Eq. (25) becomes

$$\begin{aligned}\langle \psi_{m'} | \psi_m \rangle &= \sum_{n=0}^{N-1} \delta_n e^{i\alpha\varepsilon_n} \\ &= \frac{1}{i\alpha} \sum_{n=0}^{N-1} e^{i\alpha\varepsilon_n} \left[(e^{i\alpha\delta_n} - 1) - \sum_{j=2}^{\infty} \frac{(i\alpha\delta_n)^j}{j!} \right] \\ &= -\frac{1}{i\alpha} \sum_{n=0}^{N-1} \sum_{j=2}^{\infty} e^{i\alpha\varepsilon_n} \frac{(i\alpha\delta_n)^j}{j!} \end{aligned} \tag{B.2}$$

Again making the assumption that $E_n \sim n^c$ for large n, we can show that the magnitude of this sum is $O(k/N^{2c})$.

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