How cheap can mechanics' first principles be?

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Abstract

It is fashionable today to explain various phenomenological laws as emergent properties of appropriate collective behavior. Here, we argue that the very fundamental principles of physics show symptoms of being emergent properties, and thus beg for further reduction.

1 Introduction

One often speaks, generically, of 'the laws of physics'. The physicist, however, is well aware that different kinds of laws have a different status, and according to their status are meant to play a different role in both theory and applications. The major status categories are roughly as follows.

- *Analytical mechanics.* Here we have physics' "constitution"—the principles of classical mechanics, relativity, quantum mechanics. When we say, "Let's consider a Hamiltonian of this form," we do not pretend that a physical system governed by such law actually exists; we merely imply that the law would not be struck down by physics' supreme court as "unconstitutional."

- *Fundamental processes.* Here we have those physical interactions that are actually observed and that presumedly belong to nature's most fundamental repertoire. They are the "op-codes" (using Margolus' metaphor[5]) which the Supreme Architect actually decided to include in physics' "machine language." We tentatively assume that, as in the design of a computer chip, other choices of op-codes were potentially available and could have been equally effective.

  Of course, some "grand unified theory" may later show that what appeared to be independent choices at the op-code level are actually forced consequences of a single master choice. Moreover, we may realize that what we thought was a primitive op-code is actually implemented as a higher-level construct—a "subroutine call." But we are all familiar with this kind of issues from experience with man-made worlds.[1]

- *Statistical mechanics.* Here we have laws that emerge out of the collective behavior of a large number of elements. The quantities involved in these laws may not even be meaningful for individual systems or experiments. Intuitively, one may expect that almost every detail of the microscopic interactions will be washed out by macroscopic averaging; only features that are supported by a definite conspiracy (such as a particular symmetry or conservation law) will bubble up all the way to the macroscopic surface and emerge as recognizable statistical laws.

In the past few decades, an enormous range of complex physical phenomena have been successfully explained as inexorable statistical-mechanical consequences of known fundamental processes or plausible stylizations thereof. Without doubt, the reduction of phenomenology to fundamental processes via statistical mechanics is today one of the most productive paradigms (cf. Kuhn[4]) of mathematical physics. Explaining the texture of mayonnaise has become a likely subject for ten articles in Physical Review, and no one would be surprised if its mathematics turned out to be isomorphic to that needed to explain the fine structure of quarks.

This work on collective phenomena has revealed principles that appear to have a universal and fundamental character not unlike that of the principles of mechanics. In this paper, we shall turn the tables and ask, "Are perhaps the very *principles of mechanics* so universal and fundamental just because they are emergent aspects of an extremely fine-grained underlying structure, and thus chiefly *mathematical* rather than *physical* in contents?"

A coin, no matter what its composition, shape, or tossing technique, can be characterized by a single real parameter k such that over a large number of trials it will come up heads close to a fraction k of the time. The existence of such a parameter is not usually regarded as a property of our physical world *per se*—a choice made by God when establishing the laws of nature; rather, it is seen as a mathematical consequence of almost any choice about physics God could have made.

In the same vein, one would like to ask whether, for instance, the laws of mechanics are symplectic because God explicitly decided to make them so, or whether this symplectic character automatically follows out of virtually any reasonable choice of fine-grained first principles. Similarly, can one think of simple ground rules for physics whereby relativity would appear as little surprising as the law of large numbers?

[1] The choice of op-codes for, say, the IBM/360 family of computers reveals strong constraints of economy, consistency, and completeness. And in the cheapest models of this family many of the op-codes documented in the machine-language manual are emulated by software traps rather than directly implemented in hardware; the timing may be different, but the logic is identical.

In this paper, we shall give some circumstantial evidence that questions of the above kind are scientifically legitimate and intellectually rewarding. Namely, we'll look at a number of physical concepts that are usually regarded as primitive, and in each case we'll show a plausible route for reduction to much simpler concepts.

2 Continuity

Both in the classical and the quantum description, the state of a physical system evolves as a *continuous* function of time. In mathematics, it is well known that certain discrete constructs (e.g., the distribution of prime numbers, $\pi(n)$) can be approximated by continuous ones (in this example, the Riemann function $R(x)$). However, continuity does not *invariably* emerge from discreteness through some universal and well-understood mechanism, so that, when it does, we are justified in asking *why*. Once we understand the reasons in one case, we may hope to derive sufficient conditions for its emergence in a more general situation. Here we'll give an example of sufficient conditions in a kinematical context.

Consider an indefinitely extended two-dimensional lattice of spacing λ, having a 1 ("particle") or a 0 ("vacuum") at each site, as in Figure 1a. As we move, say, along the x axis, the microscopic *density function* $\rho(x,y)$ will display the discontinuous behavior of 1b.

Figure 1: (a) Particles on a lattice. (b) Density plot along a line $y=$ const.

Let us define a whole sequence ρ_n of new density functions, with $\rho_n(x,y)$ denoting the average density over the square window of side $n\lambda$ centered at x,y. For example, ρ_3 can take any of the 10 values $0,\frac{1}{9},\frac{2}{9},\ldots,\frac{8}{9},1$ (Figure 2a). However, as x increments by λ—and the corresponding window slides one lattice position to the right—ρ_3 cannot arbitrarily jump between any two of these values; the maximum size of a jump is $1/3$ (Figure 2b).

In general, while ρ_n depends on the number of particles contained in the entire window (*volume* effect), the change $\Delta\rho_n$ corresponding to $\Delta x=\pm\lambda$ depends only on the number of particles swept by the edge of the window

Figure 2: (a) A 3×3 window centered on x, y. (b) Average-density plot, as the center of the window moves along a line $y = \text{const}$.

(*surface* effect); thus,

$$|\Delta\rho_n| \leq \frac{n}{n^2} = \frac{1}{n}, \tag{1}$$

and

$$\lim_{n\to\infty} \Delta\rho_n = 0. \tag{2}$$

If now we let the lattice spacing λ decrease in the same proportion as n increases (so that the area of the window remains constant), in the limit as $n \to \infty$ the sequence ρ_n converges to a *uniformly continuous* function of x, y.

The above considerations involving a static configuration of particles are trivial. Now, let us introduce an arbitrary discrete dynamics (τ will denote the time spacing between consecutive states), subject only to the following constraints

- *Locality*. The state of a site at time $t + \tau$ depends only on the state at time t of the neighboring sites.
- *Particle conservation*. The total number of particles is strictly conserved.

In one time step, only particles that are lying next to the window's border can move in or out of the window: much as that on x, the dependency of ρ_n on t as well is a surface effect. If in taking the above limit we let the time spacing τ shrink in the same proportion as the lattice spacing λ, so as to leave the "speed of light" (one site per step) constant, the sequence $\rho_n(x, y; t)$ converges to a uniformly continuous function of t.

Remark that, if either locality or particle conservation did not hold, ρ_n as a function of *time* would not, in general, converge to a definite limit. Thus, we have characterized a situation where the emergence of a continuous dynamics is reducible to certain general properties of a (conceptually much simpler) underlying fine-grained dynamics.

Is that the store where physics buys continuity? Who knows—but Occam would say it's a good bet!

3 Variational principles

In order to explicitly construct the evolution of an arbitrary dynamical system over an indefinitely long stretch of time one needs laws in *vectorial* form. In the time-discrete case, a vectorial law gives the *next state* u_{t+1} of the system as a function of the *current state* u_t,

$$u_{t+1} = F u_t; \tag{3}$$

though in many cases of interest F can be captured by a more concise algorithm, full generality demands that F be given as an exhaustive lookup table, since its values for different values of u can in principle be completely arbitrary.

In the continuous case, a vectorial law gives the *rate of change* of the current state $u(t)$,

$$\frac{d}{dt} u = f u(t); \tag{4}$$

where f can be thought of as a lookup table having a continuum of entries rather than a discrete set of entries.

Vectorial laws of the form (3) or (4) are very general, and can be used to describe systems that have nothing to do with physics. Only a subset of such laws, characterized by particular constraints on the form of F or f, will describe admissible physical systems. Thus, as long as we restrict our attention to physical systems, the lookup tables used in (3) or (4) have less than maximal algorithmic entropy, and can in principle be compressed. For example, in a Hamiltonian system with one degree of freedom, the state can be written as an ordered pair $u = \langle q, p\rangle$ in such a way that, instead of two lookup tables f and g, as in the hypothetical general case

$$\begin{cases} \dfrac{d}{dt} q = f\langle q, p\rangle \\ \dfrac{d}{dt} p = g\langle q, p\rangle \end{cases}, \tag{5}$$

one only needs a *single* lookup table H, as in the well-known Hamilton equations

$$\begin{cases} \dfrac{d}{dt} q = \dfrac{d}{dp} H\langle q, p\rangle \\ \dfrac{d}{dt} p = -\dfrac{d}{dq} H\langle q, p\rangle \end{cases}. \tag{6}$$

This compression factor of 2 (of $2n$, for n degrees of freedom) is attained at a cost. To obtain the current value of dq/dt it is no longer enough to look at a single entry of a table, as in (5); in fact, one has to determine the *trend* of H for variations of p in the vicinity of the current value of $\langle q, p\rangle$, and this entails looking up a whole range of entries. Compressed data save memory space, it is true, by entail more computational work.

3.1 $T = dS/dE$ for almost all systems

In general, there appear to be strong constraints on the form of admissible physical laws; these constraints are best expressed by variational principles. Could it be that the actual constraints are much weaker, and that the stronger constraints that we see are the result of our way of *perceiving* these law—perhaps through heavy statistical averaging?

One of the simplest variational principles is the relation $T = dS/dE$, where T denotes period, E, energy, and S, action. Here we show that, for the *most general* class of discrete, invertible dynamical systems, the typical class element still obeys the same relation; yet in these systems the dynamics is defined by an *arbitrary* permutation of the state set! One may wonder, then, whether this relation appears in physics as a consequence of combinatorial principles of a general nature—much as the law of large numbers—rather than as the expression of physics-specific principles.

In Newtonian mechanics, consider a conservative system with one degree of freedom. Let T be the period of a given orbit of energy E, and dS the volume of phase space swept when the energy of the orbit is varied by an amount dE (Figure 3). As is well known (cf. [1]), these quantities obey the following relation

$$T = \frac{dS}{dE}. \tag{7}$$

Quantities analogous to T, dE, and dS can be defined for dynamical systems of a much more general nature. Under what conditions will the above relation still hold? We shall show that (7) is a statistical consequence of very weak assumptions on the structure of the dynamics.

Figure 3: An energy variation, dE, and the corresponding action variation, dS.

We shall consider the class $\mathcal{X}_N$ consisting of *all* discrete systems having a finite number N of states and an invertible but otherwise arbitrary dynamics. We may assume that continuous quantities arise from discrete ones in the limit $N \to \infty$. In general, relation (7) will not hold for each individual system; however, if one looks at the class as a whole, one may ask whether this relation holds approximately for most systems of the class. Alternatively, one may ask whether this relation holds for a suitably-defined "average" system—treated as a representative of the whole class.

A similar approach is widely used in statistical mechanics.[2] In our context, though, statistical methods are applied to "ensembles" in which the missing information that characterizes the ensemble concerns a system's *law* rather than its initial *state*.

The ensemble $\mathcal{X}_N$ consists of $N!$ systems—i.e., all possible permutations of N states. Systems of this kind have very little structure; nonetheless, one can still recognize in them the "precursors" of a few fundamental physical quantities. For instance, the period T of an orbit is naturally identified with the number of states that make up the orbit. Likewise, a volume S of state space will be measured in terms of how many states it contains. It is a little harder to identify a meaningful generalization of energy; the arguments presented in Section 3.2 suggest that in this case the correct identification is $E = \log T$, and this is the definition that we shall use below.

Armed with the above "correspondence rules," we shall investigate the validity of relation (7) for the ensemble $\mathcal{X}_N$.

Each system of $\mathcal{X}_N$ will display a certain distribution of orbit lengths; that is, one can draw a histogram showing, for $T = 1, \ldots, N$, the number $n(T)$ of orbits of length T (cf. Figure 4).

Figure 4: Orbit-length histogram.

If in this histogram we move from abscissa T to $T + dT$ we will accumulate a count of $n(T)\,dT$ orbits. Since each orbit contains T points, we will sweep an amount of state space equal to $dS = T\,n(T)\,dT$; thus

$$\frac{dS}{dT} = T\,n(T). \tag{8}$$

[2] For example, in an ideal gas, almost all the systems in an ensemble at a given temperature display a velocity distribution that is very close to the Boltzmann distribution; the latter can thus be taken as the "representative" distribution for the whole ensemble, even though hardly any element of the ensemble need follow that distribution *exactly*.

On the other hand, since $E = \log T$,

$$\frac{dT}{dE} = T; \tag{9}$$

hence

$$\frac{dS}{dE} = \frac{dS}{dT}\frac{dT}{dE} = T^2 n(T). \tag{10}$$

Therefore, the original relation (7) will hold if and only if the orbit-length distribution is of the form

$$n(T) = \frac{1}{T}. \tag{11}$$

Do the systems of $\mathcal{X}_N$ display this distribution?

Observe that, as N grows, the number of systems in $\mathcal{X}_N$ grows much faster than the number of possible orbit-length distributions: most distributions will occur many times, and certain distributions may appear with a much greater frequency than others. Indeed, as $N \to \infty$, almost all of the ensemble's elements will display a similar distribution. In such circumstances, the "typical" distribution is just the *mean distribution* over the ensemble, denoted by $\overline{n(T)}$.

It turns out that for $\mathcal{X}_N$ the mean distribution is exactly

$$\overline{n_N(T)} = \frac{1}{T} \tag{12}$$

for any N.

In fact, we construct a specific orbit of length T by choosing T states out of N and arranging them in a definite circular sequence. This can be done in $\binom{N}{T}\frac{T!}{T}$ different ways. To know in how many elements of the ensemble the orbit thus constructed occurs, we observe that the remaining $N-T$ elements can be connected in $(N-T)!$ ways. Thus, the total number of orbits of length T found anywhere in the ensemble is

$$\binom{N}{T}\frac{T!}{T}\cdot(N-T)! = N!\frac{1}{T}. \tag{13}$$

Divide by the size $N!$ of the ensemble to obtain $1/T$.

Thus, the typical system of $\mathcal{X}_N$ obeys relation (7). Intuitively, when N is large enough to make a continuous treatment meaningful, the odds that a system picked at random will appear to be governed by the variational principle $T = dS/dE$ are overwhelming.

3.2 Why $E = \log T$

Finite systems lack the rich topological structure of the state space found in analytical mechanics. Beside invertibility, in general the only *intrinsic*[3] structure that they are left with is the following:

[3] I.e., independent of the labeling of the points, and thus preserved by any isomorphism.

- Given two points a and b, one can tell whether b can be reached from a in t steps; in particular (for $t = 0$), one can tell whether or not $a = b$.

Thus, for instance, one can tell how many orbits of period T are present, but of these one cannot single out an individual one without actually pointing at it, because they all "look the same."

To see whether there is a quantity that can be meaningfully called "energy" in this context, let us observe that physical energy is a function E, defined on the state space, having the following fundamental properties:

1. *Conservation.* E is constant on each orbit (though it may have the same value on different orbits).
2. *Additivity.* The energy of a collection of weakly-coupled system components equals the sum of the energies of the individual components.
3. *Generator of the dynamics.* Given the constraints that characterize a particular class of dynamical systems, knowledge of the function E allows one to uniquely reconstruct, *up to an isomorphism*, the dynamics of an individual system of that class.

The proposed identification $E = \log T$ obviously satisfies property 1.

As for property 2, consider a finite system consisting of two independent components, and let a_0 and a_1 be the respective states of these two components. Suppose for a moment that a_0 is on an orbit of period 3, and a_1 on one of period 7; then the overall system state $\langle a_0, a_1\rangle$ is on an orbit of length 21, i.e., $\log T = \log T_0 + \log T_1$. This argument would fail if T_0 and T_1 were not coprime. However, for randomly chosen integers the expected number of common factors grows extremely slowly with the size of the integers themselves[7], so that approximate additivity holds almost always.

As for property 3, an individual system of $\mathcal{X}_N$ is completely identified—up to an isomorphism—by its distribution $n(T)$, and thus any "into" function of T (in particular, $E = \log T$) satisfies this property.

Note that the ensemble $\mathcal{X}_N$ consists of *all* invertible systems on a state space of size N. If we placed further constraints on the make-up of the ensemble, i.e., if we restricted our attention to a subset of systems having additional structure, some of the above arguments may cease to be valid. For example, while it is true that for large N *almost all* subensembles of $\mathcal{X}_N$ retain distribution (12), yet in a few "perverse" cases the distribution will substantially depart from $1/T$, and, if we still assume that $E = \log T$, (7) may fail to hold. Moreover, systems that were isomorphic within $\mathcal{X}_N$ may no longer be so when more structure is introduced; to allow us to tell that two systems are intrinsically different, the energy function may have to be "taught" to make finer distinctions between states than just on the basis of orbit length. But all this is besides the point we are making here; a fuller discussion of these issues will be found in [10].

3.3 Conclusions

The fact that a specific variational principle of mechanics emerges quite naturally, via statistical averaging, from very weak information-mechanical assumptions, does not tell us much about what fine-grained structure, if any, may actually underlie traditional physics; the relevant point is that we come to recognize that such a principle happens to be of the right form to be an emergent feature. When we see a Gaussian distribution in a sequence of heads and tails, we can't really tell *what* coin is being tossed; but conceptual economy will make us guess that somebody is tossing *some kind* of coin, rather than concocting the sequence by explicit use of the Gaussian function.

4 Relativity

The fact that the physics of flat spacetime is Lorentz- rather than Galilean-invariant is usually treated as an independent postulate of physics, much as Euclid's fifth axiom in geometry. In other words, God could have chosen differently; Lorentz-invariance has to be acknowledged, not derived.

However, if we look at the most naive models of distributed computation, we see that Lorentz-invariance naturally emerges as a *statistical* feature, and admits of a very intuitive information-mechanical interpretation. Much as in the previous section, we do not want to claim that this is the way relativity comes about in nature; we just want to stress that the mathematics of relativity happens to lie in one of those universality classes that arise from collective phenomena.

4.1 Orientation

Consider the two-dimensional random walk on the x, y lattice. At the microscopic level, this dynamics is not rotation-invariant (except for multiples of a quarter-turn rotation); however, invariance under the continuous group of rotations *emerges* at the macroscopic level (Fig. 5). In fact, for $r^2 = x^2 + y^2 < t$ and in the limit as $t \to \infty$, the probability distribution $P(x, y; t)$ for a particle started at the origin converges to

$$\frac{1}{2\pi t} e^{-\frac{1}{2}\frac{x^2+y^2}{t}}, \tag{14}$$

i.e., depends on x and y only through $x^2 + y^2 = r^2$.

Now, there is a strict formal analogy between a *circular rotation* by an angle ϕ in the x, y plane and a Lorentz transformation with velocity β in the t, x plane—which can be written as a *hyperbolic rotation* by a rapidity $\theta = \tanh^{-1} \beta$:

$$\begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} t' \\ x' \end{bmatrix} \tag{15}$$

Figure 5: In the two-dimensional random walk on a lattice, circular symmetry naturally emerges at the macroscopic level out of a square-symmetry microscopic law.

Riding on this analogy, one may hope to find a microscopic dynamics on the t, x lattice for which Lorentz invariance (which is out of the question at the microscopic level) would emerge at the macroscopic level.

Let's look first at the one-dimensional random walk on a lattice, with probability p of moving to the right and $q = 1 - p$ of moving to the left. For $p = q = 1/2$, the evolution of the resulting binomial distribution is characterized, macroscopically, by a mean $\mu = 0$ and a standard deviation $\sigma = \sqrt{t/4}$ (Figure 6a).

Figure 6: (a) Symmetric random walk ($p = 1/2$). (b) Asymmetric random walk ($p = 3/4$); note that, as the center of mass picks up a speed $\beta = p - q$, the rate of spread goes down by a factor $1 - \beta^2$.

In general, $\mu = (p - q)t$. If we shift the parameter p away from its center value of $1/2$, the center of mass of the distribution will start moving at a uniform velocity $\beta = p - q$. Let's try to offset this motion by a Galilean transformation

$$x = x' + \beta t'. \tag{16}$$

Macroscopically, the new system will evolve, in the new frame, just as the old system did in the old frame—except that now $\sigma = \sqrt{pqt} = \sqrt{(1 - \beta^2)t/4}$, so that the diffusion will appear to have *slowed down* by a factor $1 - \beta^2$ (Fig. 6b).

Intuitively, as some of the resources of the "random walk computer" are shifted toward producing coherent macroscopic motion (uniform motion of the center of mass), *fewer resources will remain available* for the task of producing incoherent motion (diffusion). Thus, we get a slowdown reminiscent of the Lorentz-Fitzgerald "time expansion." In the present situation, however, the slowdown factor is $1-\beta^2$, related to, but different from, the well-known relativistic factor $\sqrt{1-\beta^2}$; the transformation that will restore invariance of the dynamics in this case is a Lorentz transformation followed by a scaling of both axes by a further factor $\sqrt{1-\beta^2}$.

4.2 A Lorentz-invariant model of diffusion

In the above example, when we tried to offset by a *Galilean* coordinate transformation the consequences of a transformation of the dynamical parameter p, we noticed that "proper time," as measured by σ, was not independent of μ. Time as well as space needed to be transformed in order to have the desired dynamical invariance. However, the appropriate transformation was not simply a Lorentz transformation.

But neither were we following the standard procedures of relativity. Fact is, with dynamical *parameters* we are barking up the wrong tree. What relativity says is that a certain kind of transformation of the spacetime coordinates (Lorentz transformation) can always be offset by an appropriate transformation of the dynamical *variables*. We shall now present a lattice-gas model of diffusion that has the same macroscopic phenomenology as the random walk, but is microscopically deterministic and reversible. Unlike the random walk, changes in the macroscopic parameters μ and σ arise in this model from changes in the initial distribution of microscopic *states*, rather than by tampering with the microscopic *laws*. This model is exactly Lorentz-invariant in the continuum limit, i.e., as the lattice spacing λ goes to zero.

Let us consider a one-dimensional cellular automaton having the format of Fig. 7a. This is a regular spacetime lattice, with a given spacing λ (lattice units per meter). The arcs represent *signals* traveling at unit speed (the "speed of light"); the nodes represent *events*, i.e., interactions between signals. If one of the possible signal states, denoted by the symbol 0, is interpreted as denoting the *vacuum*, the remaining states can be interpreted as *particles* traveling on fixed spacetime tracks (the arcs) and interacting only at certain discrete loci (the nodes). Such a system can be thought of as a *lattice gas* (cf. [3, 8]).

Here, we will allow no more than one particle on each track. When two particles collide, each reverses its direction (Fig. 7b). As long as particles are identical (say, all black), this reversal is indistinguishable from no interaction (Fig. 7c).

Now let us paint just one particle red (in which case the reversal *does* make a difference), and study the evolution of its probability distribution $\rho(x;t)$ when

Figure 7: (a) One-dimensional lattice, unfolded over time. The tracks, with slope ± 1, indicate potential particle paths; the nodes indicate potential collision loci. (b) Bouncing collision. (c) No-interaction collision.

both right- and left-going particles are uniformly and independently distributed with linear density (particles per meter) $s = n/\lambda$—where n is the lattice *occupation density* (particles per track).

For fixed s, as $\lambda \to 0$ (continuum limit) $\rho(x;t)$ converges to the solution of the *telegrapher's equation*

$$\partial_{tt}\rho = \partial_{xx}\rho - 2s\partial_t\rho. \tag{17}$$

The latter distribution, in turn, converges to the solution of the diffusion equation

$$\partial_t\rho = \frac{1}{2s}\partial_{xx}\rho \tag{18}$$

in the same circumstances (i.e., $t \to \infty$, $|x| < \sqrt{t}$) as the binomial distribution does. We shall now introduce the freedom to independently vary the densities s_+, s_- of, respectively, right- and left-going particles; as a consequence, the red particle's distribution's center of mass will drift, and its diffusion rate will be affected too—much as in the asymmetric random walk case. However, this time we have strict Lorentz-invariance (in the continuum limit): to every Lorentz transformation of the coordinates, $t, x \mapsto t', x'$, there corresponds a similar linear transformation of the initial conditions, $s_+, s_- \mapsto s'_+, s'_-$, that leaves the form of ρ invariant. (Indeed, the telegrapher's equation is just another form of the Klein-Gordon equation used in relativistic quantum mechanics.)

Lorentz invariance emerges in a similar way for a much more general class of dynamics on a lattice, as explained in [9]; more generally, features qualitatively similar to those of special relativity appear whenever *fixed* computational resources have to be apportioned between producing the inertial motion of a macroscopic object as a whole and producing the internal evolution of the object itself (cf. [2]). Thus, we conjecture that special relativity may ultimately be derived from a simpler and more fundamental principle of *conservation of computational resources.*

4.3 General relativity

The spacetime lattice in which the particles of the above example move and interact can be thought of as a uniform combinational network—the simplest kind of parallel computer. Recall, however, that Lorentz invariance was achieved in the limit of a vanishingly fine lattice spacing, while holding the density (particles *per meter*) constant. In this limit, then, the occupation number (particles *per track*) goes to zero; this corresponds to a vanishing utilization of the network's computing resources. By the time Lorentz-invariance emerges, the model has become useless as a numerical computation scheme.

In an attempt to trade accuracy for computational efficiency, suppose we start backing up a little from limit, i.e., we consider a network with a fine, but not *infinitely* fine, spacing. As the network becomes coarser, the number of tracks available to the same number of particles decreases, and thus the occupation number increases. When this number significantly departs from zero, the macroscopic dynamics will start deviating from special relativity.

Is this really an unfortunate state of affairs? After all, we know that physics itself starts deviating from special relativity when one dumps more and more matter in the same volume. Are we witnessing the emergence of general relativity? Indeed, the slowdown of the macroscopic evolution brought about, in models of the above kind, by the "crowding" of the computational pathways, is strikingly analogous to the proper-time dilation that, in physics, is brought about by the gravitational potential.

Without more comprehensive models, precise interpretation rules, and quantitative results, any claims that the present approach might have anything to do with modeling general relativity is, of course, premature. But it is legitimate to ask whether fine-grained computation in uniform networks has at least the right kind of internal resources for the task. In other words, is the emergence plausible, in such systems, of a dynamics of spacetime analogous to that described by general relativity? and how could it come about?

Let us start with a metaphor. On a strip of blank punch tape we can record information at a density of, say, ten characters per inch. What if we could only avail ourselves of *used* tape, found in somebody's wastebasket? Knowing the statistics of the previous usage, one can devise appropriate group encoding techniques and error correcting codes so as to make such a used tape perfectly adequate for recording new information (cf. [6])—at a lower density, of course, i.e., up to the maximum density allowed by Shannon's theorems for a noisy channel. The *proper length* of the tape, defined in terms of how many characters *we* can record on it, will be less than that of blank tape, by a factor that will depend on how heavy the original usage was. If the tape is sufficiently long, its statistical properties may significantly vary from place to place, and we may want to adapt our encoding strategy to the local statistics—yielding a *proper-length* metric that varies from place to place.

Let us extend the above metaphor from the domain of information *statics* to that of information *dynamics*. Consider, for example, a programmable gate array having a nominal capacity of, say, 10,000 gates. An inventor designs a clever arcade game that takes full advantage of the chip's "computing capacity," and asks the VLSI factory to produce a million copies of it. The game turns out to be a flop, and the programmed chips get thrown in the waste basket. What is the effective "computing capacity" of these chips from the viewpoint of the pennyless but undaunted hacker that finds them? How many of these chips would he have to put together in order to construct *his own* arcade game, and how many clock cycles of the original chip would he have to string together to achieve a usable clock cycle for his game? What in the new game is simply the toggling of a flip-flop may correspond, in the underlying original game, to the destruction of a stellar empire. For the new user, *proper time* will be measured in terms of how fast the evolution of *his* game can be made to proceed.

For a macroscopic scavenger, the individual hole positions in a punched tape or the individual gates in an electronic circuit blend into a continuum, locally characterized by a certain effective density of information-storage capacity and a certain effective density of information-processing capacity. These densities reflect the constraints that the local "degree of congestion" of the computing resources sets on any "further incremental usage" of these resources. Thus, if *length* and *time* measure, respectively, the effective information-storage and -processing capacities available to macroscopic epiphenomena, a metric and a dynamics of *curved* spacetime naturally emerge out of a *flat*, uniform computing network.

4.4 Conclusions

Quantitative features of special relativity and at least qualitative features of general relativity emerge quite naturally as epiphenomena of very simple computing networks. Thus, relativity appears to be of the right form to be an emergent property, whether or not *that* is the way it comes about in physics.

5 General conclusions

Many of what are regarded as the most fundamental feature of physics happen to have the right form to be emergent features of a much simpler fine-grained dynamics.[4]

A century and a half ago, most people were happy with the idea that the cell was a bag of undifferentiated "protoplasm" governed by some irreducible "vital force." The behavior of the cell was obviously very rich, but few people dared

[4]Even *invertibility*—perhaps the most strongly held feature of microscopic physics—can quite naturally emerge out of an underlying noninvertible dynamics. We are going to discuss this topic in a separate paper.

to ascribe it to much finer-grained internal machinery, explicitly built according to immensely detailed blueprints.

Today we know for sure about the existence of such machinery and such blueprints. Besides molecular genetics, chemistry and nuclear physics provide further case histories where complex behavior was successfully reduced to simpler primitives on a grain a few orders of magnitude finer.

For a physicist, the possibility of explanation by reduction to simpler, smaller structures is of course one of the first things that comes to mind. The point of this paper is that one should look for such possibility not only to explain specific phenomenology, but also to re-examine those general principles that are so familiar that no "explanation" seems to be needed.

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References

[1] ARNOLD, Vladimir, *Mathematical Methods of Classical Mechanics*, Springer-Verlag (1978).

[2] CHOPARD, Bastien, "A cellular automata model of large scale moving objects" (1989), submitted to *J. Phys. A*.

[3] HARDY, J., O. DE PAZZIS, and Yves POMEAU, "Molecular dynamics of a classical lattice gas: Transport properties and time correlation functions," *Phys. Rev.* **A13** (1976), 1949–1960.

[4] KUHN, Thomas, *The Structure of Scientific Revolutions*, 2nd ed., Univ. of Chicago Press (1970).

[5] MARGOLUS, Norman, "Physics and Computation" (Ph. D. Thesis), *Tech. Rep. MIT/LCS/TR-415*, MIT Laboratory for Computer Science (1988).

[6] RIVEST, Ronald, and Adi SHAMIR, How to reuse a "Write-Once" memory, *Info. and Control* **55** (1982), 1–19.

[7] SCHROEDER, Manfred, *Number Theory in Science and Communication*, 2nd enlarged edition, Springer-Verlag (1986)

[8] TOFFOLI, Tommaso, and Norman MARGOLUS, *Cellular Automata Machines—A new environment for modeling*, MIT Press (1987).

[9] TOFFOLI, Tommaso, "Four topics in lattice gases: Ergodicity; Relativity; Information flow; and Rule compression for parallel lattice-gas machines," *Discrete Kinetic Theory, Lattice Gas Dynamics and Foundations of Hydrodynamics* (R. Monaco *ed.*), World Scientific (1989), 343–354.

[10] TOFFOLI, Tommaso, "Analytical mechanics from statistics: $T = dS/dE$ holds for almost any system," *Tech. Memo MIT/LCS/TM-407*, MIT Laboratory for Computer Science (August 1989) .