Problem 1: Consider a monocular system for estimating the orientation of a planar surface based on an image of a circle drawn on that surface. If we view the plane “straight on”, the circle will be imaged as a circle (by “straight on” we mean that the optical axis is perpendicular to the plane, and the optical axis passes through the center of the circle). We get an ellipse in the image if the plane is tilted so that its normal vector is no longer parallel to the image plane. From the eccentricity of the ellipse we can determine how much the plane is tilted relative to the “straight on” orientation.

To simplify matters, we here consider orthographic instead of perspective projection. Imagine parallel rays perpendicular to the image plane arriving from a very distant light source. A circular disk is suspended somewhere above the image plane blocking some of the incident light. The result is an elliptical shadow in the image. The angle between the image plane and the plane containing the circular disk is $\theta$.

(a) Express the ratio of minor axis to major axis, $b/a$, in terms of the angle $\theta$.

(b) The eccentricity $e$ of an ellipse is defined by $b^2 = a^2(1 - e^2)$. Express the eccentricity of the ellipse in the image in terms of the angle $\theta$.

(c) Image measurements are hard to make accurately. If we are to estimate $\theta$ based on the ratio of $b$ to $a$, then we should also consider the effect of small errors in the measurement of $b$ (assume $a$ is known
accurately). What is the relationship between small errors in measurement $\delta b$ and corresponding errors in the estimated orientation of the plane $\delta \theta$? Give an expression for error sensitivity $d\theta/db$ as a function of the angle $\theta$.

(d) If you had to select a good position for a camera in a monocular visual wheel alignment system based on imaging the circular rim of the wheel, would you put the camera somewhere along the line passing through the axis of the wheel?

**Problem 2:** A simple planar binocular stereo system has a baseline of length $2b$ and two 'cameras' measuring the angles $\theta_1$ and $\theta_2$ between rays to objects in the world and the baseline. We erect a coordinate system with the $x$-axis lined up with the baseline and the origin at the midpoint of the baseline. The $z$-axis is perpendicular to the $x$-axis.

(a) Show that the intersection of the two rays lies at

$$x = b \frac{\sin(\theta_1 + \theta_2)}{\sin(\theta_1 - \theta_2)} \quad \text{and} \quad z = 2b \frac{\cos \theta_1 \cos \theta_2}{\sin(\theta_1 - \theta_2)}$$
(b) Show that
\[ z \approx 2b \frac{1}{\sin(\theta_1 - \theta_2)} \]
when \( \theta_1 \approx 0 \) and \( \theta_2 \approx 0 \). Hence depth \( z \) is inversely proportional to disparity \((\theta_2 - \theta_1)\) (and proportional to the baseline \(2b\)).

(c) The measurements of ray directions are never perfectly accurate. Estimate the errors in \( z \) resulting from errors \( \delta_1 \) in measuring \( \theta_1 \), and \( \delta_2 \) in measuring \( \theta_2 \). Show that the absolute error in estimating depth grows as \( z^2/b \). Consequently the relative error (i.e. \( \delta z/z \)) grows as \( z/b \) (Hint: differentiate \( z \) w.r.t. \( \theta_1 \) and \( \theta_2 \)).

Problem 3: In an image of a rectangular raised ‘button’ the brightness of the indicated regions is as follows: A has grey value 159, B has grey value 125, and C has grey value 141. We are told that the bevelled edges of the button are inclined 45 degree with respect to the plane of the background.

![Button Image]

(a) Consider a coordinate system in the plane of the background, with \( x \) running to the right and \( y \) upwards. Let the \( z \) axis be perpendicular to the background plane. Write the normals of each of the regions A, B, and C as vectors with three components.

(b) Next, assume that the surface has reflecting properties that closely match Lambert’s ‘law’, that is, that brightness is proportional to the cosine of the incident angle. Using the three measurements of brightness, find a unit vector in the direction of the light source.

Problem 4: Consider a perspective image of a rectangular brick. Three vanishing points can be obtained by intersecting extended image lines corresponding to parallel edges on the object.
(a) Suppose that the vanishing points are at \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) in the image plane. Suppose that the center of projection is at \( \mathbf{r} \) above the image plane. Write down three (quadratic) equations involving the vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \) and \( \mathbf{r} \), based on the expected angles between lines from the center of projection to the vanishing points in the image plane.

(b) Eliminate the quadratic terms to obtain three (redundant) linear equations in \( \mathbf{r} \).

(c) Consider the triangle formed in the image plane by connecting the vanishing points. Show that the principal point \( \mathbf{r}' \) (foot of the perpendicular from the center of projection onto the image plane) lies at the
intersection of the perpendiculars dropped from the three vertices onto the opposite sides. (Hint: \( r = r' + f\hat{z} \) where \( f \) is the principal distance, that is the height of the center of projection above the image plane, and \( \hat{z} \) is a unit vector perpendicular to the image plane).

(d) Consider the special case when the vanishing points form an equilateral triangle. Show that the principal distance \( f \) equals \( kl \), where \( l \) is the length of the sides. What is the value of \( k \)?

**Problem 5:** We can think of an image as a surface in three dimensions, where the height above the \( x\)-\( y \) plane represents the brightness \( E(x,y) \) at the point \((x,y)\). At each point on this surface we can define \( E_x(x,y) \) and \( E_y(x,y) \) to be the slopes in the \( x \)- and \( y \)-directions respectively. If we move \( \delta x \) in the \( x \)-direction and \( \delta y \) in the \( y \) direction then the change in 'height' is clearly

\[
\delta E = E_x \delta x + E_y \delta y
\]

If instead the surface (i.e. image) moves, then the change in height (i.e. brightness) at a particular point will be that quantity negated.

(a) Use this observation to derive the ‘constant brightness constraint’:

\[
u E_x + v E_y + E_t = 0
\]

(b) Find the image velocity \((u, v)\) that best fits the observed brightness gradients \( E_x(x,y), E_y(x,y) \) and \( E_t(x,y) \) in the least squares sense over a rectangular image region (Hint: integrate the error squared over the region).

(c) Consider an image where brightness can be expressed in the form \( E(x,y) = f(ax + by) \) everywhere within the image region. Describe the appearance of such an image. What is the relationship between \( E_x \) and \( E_y \)? Can the above method be used to recover image motion?