Problem 1a: Consider a polyhedral object with a Lambertian surface. A trihedral corner is formed by the intersection of three planes with surface normals $\hat{n}_1$, $\hat{n}_2$, and $\hat{n}_3$. The object is illuminated by a distance point source in direction $\hat{s}$. Suppose now that the object is turned just so that the edges meeting at the corner are not visible in the image because the brightnesses of the images of the three facets match.

(a) Express the direction to the light source in terms of the unit normals of the three facets of the polyhedral object.

(b) Show that the source direction can be expressed as a weighted sum of the three surface normal vectors in the special case where the three facets are at right angles to one another (e.g. the corner of a cube) and the weights are dot-products of source direction and surface normals.

Problem 1b: Consider a “corrugated” Lambertian surface where

$$z(x, y) = A \cos\left(\frac{2\pi x}{\lambda}\right)$$

Assume that a distant viewer is vertically above the surface (i.e. $\hat{v} = \hat{z}$), that a distant light source lies in a direction $\hat{s}$ different from the viewing direction, and that the undulations are of low enough amplitude that the surface slope is everywhere small (i.e. much less than one). Show that the brightness of the surface also has a “corrugated” character, but that it is shifted in phase relative to the undulations in height.

(a) Can you recover the amplitude of the undulations on the surface from the amplitude of the undulations in brightness?

(b) Where would you place the light source to get the most accurate estimate of surface undulation amplitude in the presence of image brightness measurement noise?

(c) How would you pick the wavelength $\lambda$ of the surface undulations to get the maximum image brightness variations with fixed amplitude $A$ of surface undulations?

Problem 2: Let’s for the moment assume that the moon is made of green cheese, and that green cheese obeys Hapke’s law (i.e. brightness proportional to $\sqrt{\cos \theta_i / \cos \theta_e}$). Furthermore let’s assume that the ‘albedo’ is
constant, and finally that the surface is actually spherical (that is, there are no mountains or ‘oceans’).

Sketch the image of the moon under these simplifying assumptions. Indicate on your sketch the plane containing the light source, the moon and the viewer (i.e. the ecliptic plane). Do not assume that it is full moon — that is, let the viewer lie in a direction \( \hat{v} \) different from that of the light source \( \hat{s} \).

What shapes do the isophotes (lines of constant brightness) have in the image? Do the isophotes intersect? If so, where? How does the brightness distribution differ from that we would have obtained if we had assumed that the moon was a Lambertian reflector?

Notes: Remember that in the case of a sphere, surface normals are parallel to radius vectors from the center. You may find it helpful to project the gradient space map of a Hapke surface back onto the Gaussian sphere. Or you may want to find a linear constraint on the surface normals along an isophote. Then find all the points on the sphere that satisfy this constraint. Finally project the surface of the sphere orthographically into the image plane.

Problem 3: Consider a Lambertian surface illuminated ‘straight on’ — that is, the light source lies in the same direction as the viewer.

(a) Suppose we obtain an image where
\[
E(x, y) = \frac{\sqrt{S^2 - x^2 - y^2}}{S} \quad \text{for } x^2 + y^2 < S^2
\]
Assuming that the object is rotationally symmetric about the origin, what is its shape?

(b) Suppose instead we obtain an image where
\[
E(x, y) = \frac{1}{\sqrt{1 + A(x^2 + y^2)}} \quad \text{for some constant } A
\]
Assuming that the object is rotationally symmetric about the origin, what is it’s shape?

Problems 4: A rotation of the image can be described by the equations:
\[
\begin{align*}
    x' &= +x \cos \theta + y \sin \theta \\
    y' &= -x \sin \theta + y \cos \theta
\end{align*}
\]
(a) Find the relationship between the gradient \((z_x', z_y')\) measured in the rotated coordinate system and the gradient \((z_x, z_y)\) in the original
coordinate system. Show that
\[ z''x + z''y = z''x + z''y \]
that is, the slope in the direction of steepest ascent is the same in the two coordinate systems.

(b) Suppose we have a curve on the surface defined by \((x(s), y(s), z(s))^T\), where \(s\) is a parameter that varies along the curve (not necessarily arc length). What is the slope \(q'\) of the surface measured in the direction tangent to this curve?

(c) Suppose that from brightness measurements we can determine that the magnitude of the gradient (i.e. the slope in the direction of steepest ascent) is \(g(s)\). What then is the slope \(p'\) orthogonal to the curve? Use the result developed in part (b).

(d) Describe an algorithm for producing a new curve from the given curve by taking — at each point on the curve — a small step in the direction perpendicular to the curve. To be concrete, suppose that the reflectance map is given by
\[ R(p, q) = \frac{1}{\sqrt{1 + p^2 + q^2}} \]
Show that
\[ (p')^2 = 1 - \frac{E^2}{E^2} - \frac{z^2}{x_s^2 + y_s^2} \]
where \(E\) is the brightness measured at a point on the curve, while \(x_s\), \(y_s\), and \(z_s\) are derivatives of \(x\), \(y\) and \(z\) respectively along the curve.

**Problem 5a:** Consider a photometric stereo system where surface orientation and albedo (\(\rho\)) of a Lambertian surface are to be determined using three distant point sources. Show that if the three point sources lie in a plane containing the origin (that is, the point where the object being viewed lies), then the information obtained from the three images is not independent and so it is not possible to recover both surface orientation and albedo.

In fact, find an expression for brightness at a pixel in the third image in terms of the brightness at the same pixel in the other two images. Assume that the light source positions are \(\hat{s}_1, \hat{s}_2, \) and \(\hat{s}_3\).

**Problem 5b:** Photometric stereo in the case of a Lambertian surface illuminated by two distant point sources yields two second order equations in the components of the gradient \(p\) and \(q\). According to Bezout’s theorem this means that there can be no more than \(2 \times 2 = 4\) solutions for
surface orientation. Which may seem reasonable, since two arbitrarily rotated ellipses can in fact intersect in up to four places.

Combine the two second order equation in such as way as to obtain an equation that is linear in $p$ and $q$. The solutions are now given by this linear equation and either one of the original second order equations. So what is the actual maximum number of solutions for surface orientation?

Explain the result in terms of ‘reflectance maps’ plotted on the unit sphere instead of in gradient space. Use geometrical arguments rather than analytical methods for this part of the question.