Problem 1: Consider the solid of revolution obtained by rotating an ellipse about one of its major axes. Let the major and minor axes have length \(a\) and \(b\). Suppose that the generating ellipse lies in the \(x\)-\(y\) plane and that the axis of rotation is the \(y\)-axis.

(a) Show that the curve defined by the following is an ellipse:
\[
  x = a \cos \theta \quad \text{and} \quad y = b \sin \theta.
\]

(b) Show that
\[
  \frac{ds}{d\theta} = \sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta},
\]
where \(s\) is arc length along the ellipse.

(c) Show that
\[
  \frac{1}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}} \begin{pmatrix} b \cos \theta, a \sin \theta \end{pmatrix}
\]
is a unit normal to the ellipse at the point \( (a \cos \theta, b \sin \theta)^T \).

Note that the unit normal can also be written as \( (\cos \eta, \sin \eta)^T \) in terms of the direction of the normal vector. Show that
\[
  \frac{b^2 \cos^2 \theta + a^2 \sin^2 \theta}{ab} = \frac{ab}{a^2 \cos^2 \eta + b^2 \sin^2 \eta}
\]
where \(\eta\) is the angle between the normal vector and the \(x\)-axis.

(d) Show that \( \tan \eta = (a/b) \tan \theta \) and that
\[
  \frac{d\eta}{d\theta} = \frac{ab}{b^2 \cos^2 \theta + a^2 \sin^2 \theta} = \frac{a^2 \cos^2 \eta + b^2 \sin^2 \eta}{ab}.
\]

(e) Give the curvature \(K_G\) of the generating curve, first as a function of the parameter \(\theta\), and then as a function of the angle of the normal \(\eta\).

(f) Finally, show that the Gaussian curvature of the solid of revolution can be written in the form
\[
  K = \frac{1}{a^2} \left( \frac{a^2 \cos^2 \eta + b^2 \sin^2 \eta}{a^2 b^2} \right)^2.
\]
Verify the result for the case \(a = b\).

(f) What is the extended Gaussian image \(G(\eta, \xi)\) of this object? What are the extreme values of \(G(\eta, \xi)\)?
Hint: You may need trigonometric identities like $1 + \tan^2 \alpha = \sec^2 \alpha$ and $\frac{d}{d\alpha} \tan \alpha = \sec^2 \alpha$

See the Appendix of *Robot Vision* for more.

**Problem 2:** Consider a planar curve with the curious property that its curvature is proportional to the distance from some line. That is $K_G = Ar$, where $K_G = \frac{d\eta}{ds}$, $A$ is a constant, and $r$ is the perpendicular distance from the line. At the points where the curve touches the line it is perpendicular to the line. Now imagine spinning the curve about the line. Compute the extended Gaussian image (EGI) of the resulting solid of revolution. Compare with the EGI for a torus with the same axis direction.

**Problem 3:** The Gaussian curvature of a surface $z(x, y)$ is given by

$$\kappa = \frac{z_{xx}z_{yy} - z_{xy}^2}{(1 + z_x^2 + z_y^2)^2}$$

(a) Show that for a sphere of radius $R$, the Gaussian curvature equals $1/R^2$.

A developable surface is one that can be cut open and laid out on a plane. Such a surface has zero Gaussian curvature. Examples of developable surfaces include cylinders and cones.

(b) Given the reflectance map of a developable surface, determine $z_{xx}$, $z_{xy}$, and $z_{yy}$ from the image brightness measurements and appropriate derivatives.

(c) Comment on how the additional information about the shape of the surface allows us to solve the shape from shading problem in comparison to when we do not initially know anything about the surface.

**Problem 4:** The extended circular image of a simply connected, closed, convex, planar curve is defined in a similar fashion to the extended spherical image of a three-dimensional convex object (see problem 16–7 in the textbook — but check the errata).

Find the extended circular image of an ellipse with major axis $a$ lined up with the $x$-axis and minor axis $b$ lined up with the $y$-axis. Use the parametric form $x = a \cos \theta$, $y = b \sin \theta$ and let $\eta$ be the angle of the normal vector measured from the $x$-axis in a anti-clockwise direction. Find the tangent vector $r_\theta$, where $r = (x, y)^T$. Find a normal vector. Show that $\tan \eta = (a/b) \tan \theta$. Express $\cos \eta$ and $\sin \eta$ in terms of $\cos \theta$ and $\sin \theta$. Show that the curvature of the curve is given by

$$K = \frac{ab}{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}$$
in terms of the parameter $\theta$. Show that curvature can be written in the form

$$ K = \frac{(a^2 \cos^2 \eta + b^2 \sin^2 \eta)^{3/2}}{a^2 b^2} $$

in terms of the direction $\eta$. Check your result in the special case when $a = b$. When $a \neq b$, what are the minimum and maximum values of curvature?

**Hint:** You may be able to use intermediate results from Problem 1 here.

**Problem 5:** Here we consider rotations of a tetrahedron that bring it back into alignment with itself.

(a) Give the rotations (as unit quaternions) of a tetrahedron that bring it back into alignment with itself. For concreteness, consider the tetrahedron constructed by connecting the vertices

$$ (+1, +\sqrt{2}, 0)^T $$
$$ (+1, -\sqrt{2}, 0)^T $$
$$ (-1, 0, +\sqrt{2})^T $$
$$ (-1, 0, -\sqrt{2})^T $$

You can either explicitly construct all possible rotations through $0, 2\pi/3, \pi, 4\pi/3$, or construct unit quaternions for just two rotations through $2\pi/3$ about different axes and obtain the complete rotation group by transitive closure.

(b) Suppose we tessellate a unit sphere by projecting the faces of a tetrahedron from the center of the tetrahedron and that the accumulated totals of orientations of surface elements of some object are $A, B, C, D$ in the four cells respectively. Clearly each of the above rotations permutes the totals in the four cells. Are all possible permutations of the four weights generated by the rotations of the tetrahedron?

(c) Consider now a finer tessellation, based on the icosahedron. How many ways are there of permuting the accumulated weights? How many of these permutations are actually generated by rotations of the icosahedron?